Matrices on a TI-83/83+

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At *www.unsw.adfa.edu.au/pems/news/high_school/hsc_activities.html*

- A variety of graphics-calculator activities for Years 9 and 10 — written as part of the CQTP Program.
- TI-83 programs and program information.
- *Using the TI-83/83+* — an introduction to the basic operations, suitable for Years 8–12.
- *The Graphics Screen and Accuracy* — information to help you understand the graphical and numerical limitations of a graphics calculator.
- *Coordinate Geometry on a TI-83/83+* — basic commands and a variety of problems, suitable for Years 9 and 10.
- *Population Modelling* — a variety of problems from simple exponential growth to Leslie matrices and difference equations, covering Years 7–12.
- *Calculus on a TI-83/83+* — basic commands and a variety of problems, suitable for Years 11 and 12.
- *Sequences and Series on a TI-83/83+*— basic commands and a variety of problems, suitable for Years 11 and 12.
- *Complex Numbers on a TI-83/83+* — suitable for Years 11 and 12.
- *Introduction to Complex Numbers* — complex numbers from the beginning, covering the basic operations, but set in the context of complex numbers as a mathematical structure.
- *Programming a TI-83/83+* — suitable for teachers and keen students.

1 The Basics

The basic matrix operations work just like the operations with numbers.

Let's learn by working out the matrix product $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Call the first matrix $\boldsymbol{A}$, the second $\boldsymbol{B}$.

1.1 Putting matrices in your calculator

Press [MATRX], then [▶] [▶] (or [◀]) to get to EDIT. You are now ready to edit one of the ten matrices [A] – [J]: press [1] to select [A].

Now input the order (rows × columns) and the elements of [A] — the cursor indicates what is required. Press the desired number and [ENTER]. The cursor moves on and leads you through the elements row by row. Make sure you press [ENTER] after the last element.

Press [MATRX] [◀] again, but press [2] this time to input [B].

Press [2nd] [QUIT] to return to the home screen.

1.2 Displaying matrices

You call up matrices by pressing [MATRX] and the appropriate number. For example, pressing [MATRX] [2] will display '[B]' on the screen. Pressing [ENTER] will show you the numbers in [B].

1.3 Multiplying matrices

Press [MATRX] [1] [MATRX] [2] to display '[A][B]' and press [ENTER] to display the numerical result

$$\boldsymbol{AB} = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}.$$

The result is stored in Ans (as is the result of any calculation).

If you now want to multiply by $\boldsymbol{A}$ again, i.e. to find $\boldsymbol{A}(\boldsymbol{AB})$, press [MATRX] [1] [2nd] [ANS] (on the [(–)] key) to display '[A] Ans' and then [ENTER] to give the numerical result

$$\boldsymbol{A}\big(\boldsymbol{AB}\big) = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Note: If you want to evaluate the expression $\boldsymbol{A}\big(\boldsymbol{AB}\big)$ using brackets, you need to put in the multiplication sign before the brackets, that is evaluate $\boldsymbol{A}*\big(\boldsymbol{AB}\big)$. This is because a bracket following a matrix, $\boldsymbol{A}(1,2)$ for example, is used to denote the element of a matrix, here the $1,2$ element, i.e. the number in the first row and second column.

To evaluate $\boldsymbol{A}\big(\boldsymbol{AB}\big)$, it is much simpler to omit the brackets and just evaluate $\boldsymbol{AAB}$ or $\boldsymbol{A}^2\boldsymbol{B}$.

1.4 Squaring matrices

We could also work out $\boldsymbol{A}\big(\boldsymbol{AB}\big)$ as $\boldsymbol{A}^2\boldsymbol{B}$.

Press [MATRX] [1] [x^2] [MATRX] [2] to display '[A]^2[B]' and [ENTER] to give the same result

$$\boldsymbol{A}^2\boldsymbol{B} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Integer powers of a matrix are produced the same way as with numbers (for positive integers). Only square matrices can be raised to a power.

$$\boldsymbol{A}^4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

The only negative integer 'power' that works is -1 (using the [x^{-1}] key): however, $[\mathrm{A}]^{-1}$ produces the **inverse** of (square) matrix $\boldsymbol{A}$ (see Section 3). The reciprocal of a matrix is not defined.

1.5 More involved expressions

Extensions work just as you would expect. For example, to work out $\boldsymbol{AB}+3\boldsymbol{B}$, press [MATRX] [1] [MATRX] [2] + 3 [MATRX] [2] to display '[A][B] + 3[B]', and [ENTER] to give

$$\boldsymbol{AB}+3\boldsymbol{B} = \begin{bmatrix} 7 & 11 & 15 \\ 14 & 19 & 24 \end{bmatrix}.$$

Note: If you want to evaluate an expression involving brackets, such as $\boldsymbol{A}\big(\boldsymbol{B}+\boldsymbol{C}\big)$, you need to put in the multiplication sign before the brackets, that is evaluate $\boldsymbol{A}*\big(\boldsymbol{B}+\boldsymbol{C}\big)$.

1.6 Storing matrices

If you wanted to keep the previous answer for later use, you might store it in $\boldsymbol{C}$ by pressing [STO▶] [MATRX] [3] [ENTER].

Notice that the calculator automatically makes $\boldsymbol{C}$ have the correct order.

1.7 Illegal operations

The calculator will not let you do invalid operations. For example, if you try to calculate $\boldsymbol{A} + \boldsymbol{B}$ by entering [MATRX] [1] + [MATRX] [2], you will see [A] + [B] on the screen, but pressing [ENTER] produces the message ERR: DIM MISMATCH. *Why?*

1.8 Other matrix operations

A number of operations are contained in the [MATRX] MATH menu.

- *determinant*: det([A]) $= -2$
- *transpose*: $[\mathrm{B}]^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$
- *dimension*: dim([B]) $= \{2\ 3\}$
- *'filling' a matrix*: Fill(1,[A]) produces a matrix of '1's $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
- *identity matrix*: identity(n) produces the $n \times n$ identity matrix
- *row-echelon form*: ref([A]) produces the row-echelon form of [A] (Gaussian reduction — see Section 2)
- *reduced-row-echelon form*: rref([A]) produces the reduced row-echelon form of [A] (Gauss-Jordan reduction — see Section 2)

See the TI-83 Guidebook for details on the other MATRX MATH menu items.

2 Gauss Elimination

2.1 The method

The endpoint of Gauss elimination or Gauss reduction is a matrix in **row-echelon** form, characterised by:

(a) the first non-zero element in each row is a 1 (called a **pivot**);

(b) all elements in the column below a pivot are 0.

To reduce a matrix to row-echelon form, we use **elementary row operations**.[1] The three elementary row operations are

(A) exchange two rows ($R_i \rightleftharpoons R_j$)

(B) multiply a row by a constant ($R_i \to cR_i,\ c \neq 0$)

(C) add a constant multiple of one row to another ($R_i \to R_i + cR_j,\ c \neq 0$).

Procedure

1. If the first column of the matrix is all zeros, "cross" out this column to leave a smaller matrix.

2. If the element in the first row and first column (top left element) of the matrix is 0, exchange Row 1 with another row (Operation A).

(Sometimes it is convenient to do this even if the top left element is non-zero, so as to avoid fractions in Step 3.)

3. Multiply the top row by a constant to make the first element 1 (Operation B).

4. Use Operation C to obtain '0's **below** the 1 (pivot) by adding multiples of **Row 1** to each successive row ($R_i \to R_i + cR_1,\ i = 2, 3, \ldots$).

5. "Cross out" the first row and first column to leave a smaller matrix.

6. Go back and start at Step 1 on this smaller matrix.

[1] Each row operation corresponds to the multiplication of the matrix by a corresponding elementary matrix.

Notes

- The only choice in this version of Gauss elimination is which rows to interchange in Step 2. All the other operations are prescribed.
- The solutions to the simultaneous equations described by the row-echelon matrix are the same as those of the original matrix, hence the usefulness of the method. Write down the equations corresponding to the row-echelon matrix and use back substitution to find the solutions.
- The matrix changes wherever we perform an elementary row operation, so that we cannot use equal signs between the steps. Use a $\sim$ instead. The matrices are said to be (row-)equivalent.
- We can proceed in a similar manner to obtain '0' above the pivots too, the reduced row-echelon matrix. This is called *Gauss-Jordan elimination*. The solution can be read directly from the final matrix, without the need for back substitution.

2.2 Using the calculator

The TI-83 has two built-in commands *ref* and *rref* in the MATRX MATH menu.

ref (*matrix*), where *matrix* is one of the ten matrices used by the calculator, produces the row-echelon form of *matrix* (Gauss elimination).

rref (*matrix*) produces the reduced row-echelon form of *matrix* (Gauss-Jordan elimination).

Using ►Frac in the MATH menu converts the elements to fractions where possible: for example ref ([A]) ►Frac displays the row-echelon form of [A], with elements as fractions.

The GAUSS program

The TI-83 program GAUSS just automates these procedures. After running the program, the original matrix remains in [A], with the row-echelon form stored in [D] and the reduced row-echelon form in [E].

Store the matrix in [A] using MATRX EDIT. Run the program GAUSS. At the pauses, the matrix is in

- row-echelon form — decimal version.
- row-echelon form — fraction version.
- reduced row-echelon form — decimal version.
- reduced row-echelon form — fraction version.

Press ENTER to move through the different forms and ENTER after the final form to complete the program. If you want to stop at any intermediate stage, press ON Quit. Use MATRX to recall either [D] or [E] for further calculations.

Examples: Gauss elimination

$$\begin{bmatrix} 10 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & .4 & .1 & .1 \\ 0 & 1 & -1 & -8.5 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2/5 & 1/0 & 1/10 \\ 0 & 1 & -1 & -17/2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & .\dot{3} & .1\dot{6} & .\dot{6} \\ 0 & 1 & .25 & .25 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 1/6 & 2/3 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & .25 & .25 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 5 & 1 & 3 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 Solving Systems of Linear Equations

Work through the matrix operations in the first section of these notes, if you haven't already done so.

1. Enter the matrix below carefully into [A] using MATRX EDIT.

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

2. Matrix [A] is the augmented matrix $[\boldsymbol{P}|\boldsymbol{Q}]$ for the following system of equations: in matrix form $\boldsymbol{PX} = \boldsymbol{Q}$.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 \quad &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

Solve this system

(a) using the calculator to find the row-echelon form (Gaussian reduction), and then back substitution (you have to do this).

(b) using the calculator to find the reduced row-echelon form (Gauss-Jordan reduction), from which you can read off the answer.

(c) by using $\boldsymbol{X} = \boldsymbol{P}^{-1}\boldsymbol{Q}$. Do you understand why this works?

A little manipulation on the calculator allows us to evaluate $\boldsymbol{P}^{-1}\boldsymbol{Q}$. Change [A] to the matrix $\boldsymbol{P}$ by changing the column dimension to 4 using MATRX EDIT: this chops off the last column of [A]. Store the column matrix $\boldsymbol{Q}$ in [B] and then evaluate$[A]^{-1}[B]$.

3. Solve the systems of equations over the page using each of the three methods. Check your answers by substituting them back into the equations. Can you use Method 2(c) to solve these? What happens in (b) and (c) below? Can you explain? What is $\det(\boldsymbol{P})$? What does this tell you about the inverse matrix?

(a)

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ x - 2y + 3z &= 9 \\ -x + 3y &= -4 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

(c)

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

4 Eigenvalues and Eigenvectors

Given an $n \times n$ matrix $\boldsymbol{A}$, the eigenvalues of $\boldsymbol{A}$ are the constants λ and the eigenvectors of $\boldsymbol{A}$ are the corresponding n-dimensional vectors $\boldsymbol{X}$ ($n \times 1$ matrices) satisfying the equation

$$\boldsymbol{AX} = \lambda \boldsymbol{X}.$$

The eigenvalues λ are the zeros or roots of the nth-degree characteristic polynomial

$$p(x) = \det\bigl(\boldsymbol{A} - x\boldsymbol{I}\bigr),$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix.

A nice way to find the eigenvalues on a TI-83 is graphically: use the EIGENVAL program or set Y1 = det([A]−X identity(n)), the characteristic polynomial, where you have to put in a value for n, the dimension of [A], and graph in the usual way. Use *zero* (in the CALC menu) to find the zeroes of the polynomial: these are the eigenvalues λ_i, $i = 1, 2, \ldots$.

To find the eigenvectors, we have to solve the homogeneous matrix equation

$$\bigl(\boldsymbol{A} - \lambda_i \boldsymbol{I}\bigr)\boldsymbol{X} = \boldsymbol{0}$$

for each eigenvalue λ_i found above. We do this using Gauss or Gauss-Jordan elimination, most simply done using the commands *ref* or *rref* on the matrix $\boldsymbol{A} - \lambda_i \boldsymbol{I}$ for each eigenvalue λ_i. Alternatively, we can use the GAUSS program, with $\boldsymbol{A} - \lambda_i \boldsymbol{I}$ stored in matrix [A].

Hint: If you use the program, store the original matrix $\boldsymbol{A}$ in say matrix [B]. Then the combinations $\boldsymbol{A} - \lambda_i \boldsymbol{I}$, in calculator terms [B] $-\lambda_i$identity(N), can be stored successively in [A] for use in the program.

Example

Find the eigenvalues and corresponding eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

This is a simple problem that can be done easily by hand. We use it to illustrate the calculator method, which can be used for more complicated problems.

Eigenvalues

Put $\boldsymbol{A}$ into matrix [A] in your calculator and run EIGENVAL. We don't know exactly where the zeroes of the characteristic polynomial are to be found, so we set Xmin = −5, Xmax = 5. We can change these once we see the graph.

The program plots the characteristic polynomial, here a parabola, and it is clear that the zeroes lie in the range $0 < x < 3$. ZOOM In on this region so that you can see the graph cutting the x axis. Using *zero* gives the eigenvalues as $\lambda_1 = 1$ and $\lambda_2 = 2$.

In this simple case, it is easy to show by hand that the characteristic polynomial is $p(\lambda) = \lambda^2 - 3\lambda + 2$, with zeroes 1 and 2.

Eigenvectors

$\boldsymbol{\lambda_1 = 1}$: We have to find $\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, such that $\boldsymbol{AX}=\boldsymbol{X}$, i.e. solve

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

or equivalently

$$\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

To use the GAUSS program, we need to put $\boldsymbol{A}-\boldsymbol{I}$ in matrix [A]. Since we want to use $\boldsymbol{A}$, now in [A], again later, store [A] in [B]. Then store [B]−identity(2) in [A] and run GAUSS. Alternatively execute the command ref([B]−identity(2)). Both give the row-equivalent row-echelon form of $\boldsymbol{A}-\boldsymbol{I}$ as

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeroes. There must be at least one row of zeroes (the bottom row) in all eigenvector problems.

The corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The bottom row tells us that x_2 is arbitrary, so we set $x_2 = t$, where t is any number. The top row tells us that $x_1 + x_2 = 0$, so that $x_1 = -x_2 = -t$. The eigenvector $\underset{\sim}{\boldsymbol{v}}_1$ corresponding to $\lambda_1 = 1$ is therefore

$$\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero[2]) multiples of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We usually say that this vector is the eigenvector, with the understanding that all non-zero multiples of it are also eigenvectors.

Check:

$$\boldsymbol{AX} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \boldsymbol{X}.$$

$\boldsymbol{\lambda_2 = 2}$: We have to solve $\boldsymbol{AX} = 2\boldsymbol{X}$ or $\big(\boldsymbol{A}-2\boldsymbol{I}\big)\boldsymbol{X} = \boldsymbol{0}$. Therefore, we store [B]−2 identity(2) in [A] and run GAUSS. Alternatively execute ref([B]−2 identity(2)). Both give the row-equivalent row-echelon form of $\boldsymbol{A}-2\boldsymbol{I}$ as

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $x_2 = t$, where t is any number, and $x_1 + 2x_2 = 0$, so that $x_1 = -2x_2 = -2t$. The eigenvector $\underset{\sim}{\boldsymbol{v}}_2$ corresponding to $\lambda_2 = 2$ is therefore

$$\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The eigenvector is an arbitrary (non-zero) multiple of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Check:

$$\boldsymbol{AX} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2\boldsymbol{X}.$$

[2]If $t = 0$, we obtain a zero vector, which, by definition, cannot be an eigenvector.

Exercise: Show that the eigenvalues of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

are -1, 1 and 2, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

5 Answers

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2. $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$.

3. **(a)** $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

(b) $x_1 = 1 + 2t$, $x_2 = 2 + 3t$, $x_3 = t$; t any non-zero number.

(c) No solution.

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The eigenvalues are -1, 1 and 2, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$