Sequences and Series on a TI-83/83+

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1 Introduction

These notes provide a comprehensive review of generating, displaying and graphing sequences and series on a TI-83/83+ graphics calculator. An arithmetic progression, a geometric progression and the Fibonacci numbers are used as examples. A number of questions (with solutions) illustrate the use of the calculator. Finally there are three topics that could be used as a basis for group investigation or a small project.

The TI-83 can generate sequences, sum series, and display sequence terms in a table or graph. However, we should first ask whether it makes sense to use a graphics calculator at all for sequences and series.

Certainly the first few lessons on sequences should be pencil and paper, until some of the concepts and calculations are understood, although a class activity such as that on page 8 can add variety to the early learning stages. However, having to work out terms of a sequence or series by hand eventually becomes tedious, especially those terms that are not very simple and require a calculator anyway. This becomes an impediment to further learning and exploration.

The calculator automates the process of calculating terms in a sequence or series *once it is given an appropriate definition*. It is in finding an appropriate definition that most of the thought goes — the calculator can't do this. With automatic calculation comes the ability to explore particular sequences and series, to conjecture and test, and to look at ideas such as the convergence of an infinite sequence or series.[1]

Some of the questions and investigations in Sections 4 and 5 demonstrate this extra capability when using a graphics calculator.

At this stage, it is perhaps useful for the reader to review the use of the TI-83 graphics keys, the top row, by graphing, say $y = x^3$ and generating a table of function values.

> Set a WINDOW manually first and graph $Y_1 = X^3$ over some suitable domain.
>
> Then use ZoomFit ([ZOOM] [0]) to carry out the process of finding a suitable Y scale automatically. Use both [GRAPH] and [TRACE] to generate the graph.
>
> Review line types (to the left of the function definition in [Y=]), particularly solid lines (the default) and dotted lines.
>
> For the table, use [TBLSET] to specify the starting value and increment for X. Generate a table with [TABLE].

[1] These are not esoteric beasts — the humble AP and GP continue on indefinitely.

2 Sequences

A sequence is an ordered set of numbers, usually with the numbers or terms in the sequence determined by some sort of formula.[2] For example,

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \ldots \qquad 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots \qquad 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots$$

are sequences for which we can find a formula to determine each term.

In the usual notation, a general sequence is written as

$$u_1, u_2, u_3, \ldots, u_n, \ldots,$$

where each term u_1, u_2, $u_3, \ldots$ is a number. The subscript gives the position of the term in the sequence.

The TI-83 uses the notation $\mathrm{u}(n)$ for the nth term of the sequence, rather than u_n, so the general sequence is written

$$\mathrm{u}(1),\ \mathrm{u}(2),\ \mathrm{u}(3),\ \ldots,\ \mathrm{u}(n),\ \ldots.$$

This was presumably done for ease of display, but it reinforces an important fact about sequences: a sequence is really just a function with domain the positive integers, or some subset of them.

There are two ways to give a formula for each term.

- **Recursively:** write the nth term in terms of the previous term or terms. Here we also need to know a value for the first term (or first few terms) in the sequence.

  Examples
  1. $u_2 = u_1 + 2$, $u_3 = u_2 + 2, \ldots$ or in general, $u_n = u_{n-1} + 2$. With $u_1 = 1$, this recursive formula gives the first sequence above, an arithmetic progression (AP).
  2. $u_n = u_{n-1}/2$, $u_1 = 1$ gives the second sequence above, a geometric progression (GP).
  3. $u_n = u_{n-1} + u_{n-2}$, $u_1 = 1$, $u_2 = 1$ gives the third sequence above, the famous Fibonacci numbers.

- **Explicitly:** specify the nth term as a function of n, where n takes integer values.

  Examples
  1. $u_n = 2n - 1$, $n = 1, 2, 3, \ldots$, again giving the first sequence above.
  2. $u_n = 0.5^{n-1}$, $n = 1, 2, 3, \ldots$, again giving the second sequence above.

  The Fibonacci sequence can also be defined explicitly — see Section 5.

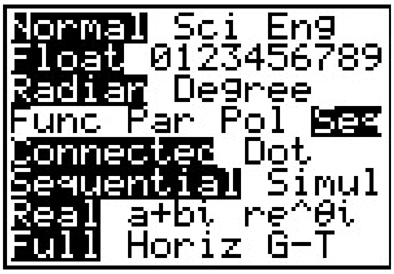
There are basically two ways on the TI-83 to generate and display terms of a sequence — *using the sequence grapher* and *using LIST commands*. Which method you use will depend on how you teach the topic. Here we'll look at both, using the above examples to illustrate the methods. There are questions to practise on in Section 4.

[2] although we can have sequences of random numbers.

2.1 Using the sequence grapher

Sequences can be defined either recursively or explicitly, displayed and graphed by the built-in sequence grapher. To select this, press MODE and, with the cursor and ENTER, select *Seq* as shown below.

The graphing mode *Seq* is just one of four possible ways to define a function on the TI-83. Because a sequence is just a function as far as the calculator is concerned, all the calculator graphing keys (top row) are relevant.



Now press the function-definition key Y= to see the three sequences u, v, w available.

On a TI-83, the nth term u(n) can be written as any combination of n, u($n-1$), u($n-2$), v($n-1$), v($n-2$), w($n-1$) and w($n-2$).

2.1.1 Arithmetic progression

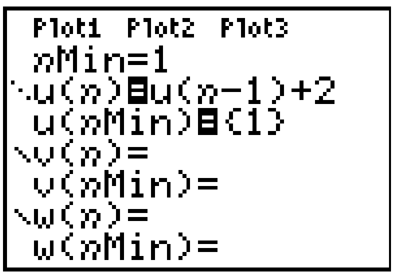
In an arithmetic progression, there is a constant difference between successive terms. In calculator notation, the recursive definition is

$$\mathrm{u}(n) = \mathrm{u}(n-1) + d,$$

where d is a constant called the common difference.

Example: $\mathrm{u}(n) = \mathrm{u}(n-1) + 2$, with $\mathrm{u}(1) = 1$.

Set this sequence up on your calculator as shown below. The independent variable here, n, is produced by the X,T,θ,n key. The sequence name u is 2nd 7. The initial value u(1) must be in a list, hence the curly brackets (on the bracket keys).

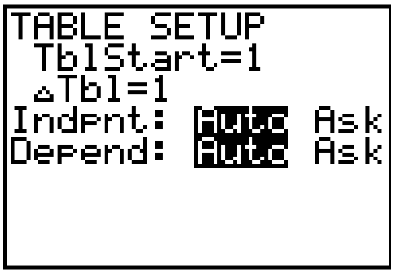


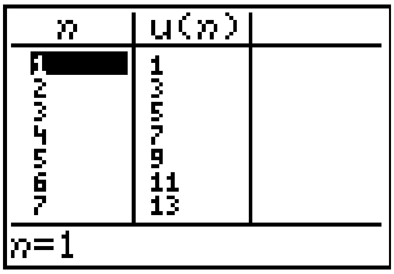
While we are in this screen, check that you have three dots to the left of the definition of u(n), indicating that only the sequence points will be plotted (no joining lines) when we graph the sequence. If not, move the cursor here and press ENTER until the three dots come up. The other two options are points joined by a normal line or a bold line.

Displaying the sequence

Here we use the TABLE feature of the calculator.

First a table 'WINDOW': press [TBLSET] ([2nd] [WINDOW]) and set TblStart=1 and ΔTbl=1.





Press [TABLE] ([2nd] [GRAPH]). Scroll down in either column to see the terms of the sequence. If you want to scroll back up past the top of the screen (after you have scrolled down), you have to be in the n column.

Another way of displaying terms of a sequence u is to type on the home screen a command of the form u (*start*, *end* [, *increment*]). For example, typing u $(1, 9, 2)$ and pressing [ENTER] will display u(1), u(3), u(5), u(7) and u(9).
increment is optional, a value of 1 assumed if it is not entered.

To display the value of just one term, just type u(n *value*), for example u(6), and press [ENTER].

Graphing the sequence

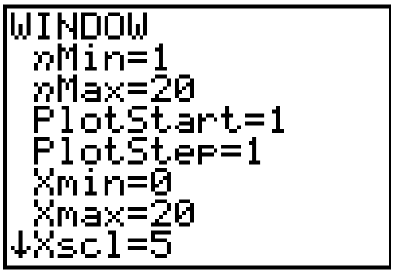
There are several ways to graph sequences, as shown when you press [FORMAT] ([2nd] [ZOOM]): there's an extra line here (the top one) in Seq MODE.

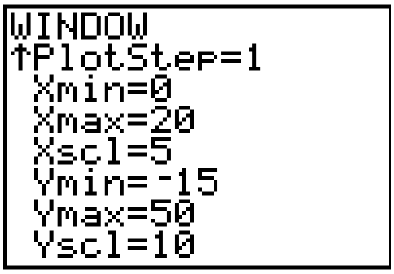
- *Time* plots u(n) against n, which is what we want here. Select it if necessary with the cursor and [ENTER].
- *Web* plots a cobweb plot, u(n) against u($n-1$). We'll look at this in Section 5.2.
- If you have more than one sequence, you can plot one sequence against another: *uv*, *vw*, *uw*.

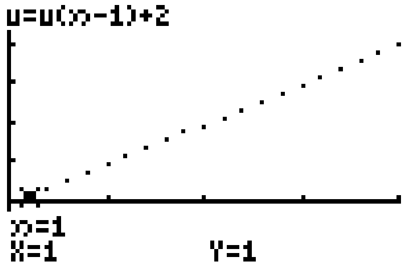
Next set a graph WINDOW: press WINDOW and you will see a few more parameters than usual. In our *Time* plot, n is plotted along the X axis, so the n range should be contained within the X range.

Set up your WINDOW as shown below and press TRACE. Use the arrow keys to move along the points of the sequence.

The value of -15 for Ymin (middle figure below) is to allow space for the coordinates at the bottom of the screen when using TRACE.







Shortcut: Specify nMax and then press ZOOM 0 (*ZoomFit*); this sets appropriate X and Y scales. You may want to change these a bit in WINDOW after the graph has been plotted The other ZOOM options, such as *ZoomIn* and *ZoomBox*, work too.

Defining the AP explicitly

The nth term for a general arithmetic progression in calculator notation is

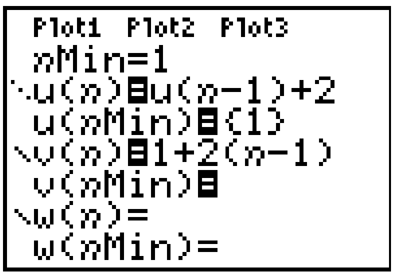
$$\mathrm{u}(n) = a + (n-1)d,$$

where a is the first term ($n = 1$) and d is the common difference.

For our example, we have $a = 1$ and $d = 2$, so that

$$\mathrm{u}(n) = 1 + 2(n-1).$$

We can compare the two definitions of the sequence by putting the nth term in sequence v, as shown below. Note that v(nMin) is not required when a sequence is defined explicitly **and should be cleared**. Plot v as a solid line so you see which sequence is being plotted.



Now press GRAPH and compare u and v. TRACE might be useful. You can also use TABLE to compare values of u and v.

2.1.2 Geometric progression

In a geometric progression, each term is a constant multiple of the previous term. In calculator notation, the recursive definition is

$$\mathrm{u}(n) = r\,\mathrm{u}(n-1),$$

where r is a constant called the common ratio or common multiplier.

Exercise: Display a table and graph the geometric sequence $\mathrm{u}(n) = 0.5\,\mathrm{u}(n–1)$, with $\mathrm{u}(1) = 1$.

For the graph, use *ZoomFit* after specifying nMax in WINDOW. You may want to go back to WINDOW after the graph is plotted to adjust some of the parameters, particularly Ymin (so the cursor coordinates don't obscure the graph), Xscl and Yscl (so there aren't too many tick marks on the axes). Press [GRAPH] or [TRACE] to return to the graph after changing WINDOW.

The nth term of a geometric progression is given *explicitly* by

$$\mathrm{u}(n) = ar^{n-1},$$

where a is the first term and r is the common multiplier.

For the sequence in the exercise above, the nth term is given by

$$\mathrm{u}(n) = 0.5^{n-1}.$$

Exercise: Put this explicit definition in sequence v and compare with the recursive definition.

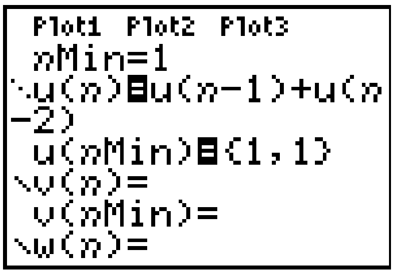
2.1.3 Fibonacci sequence

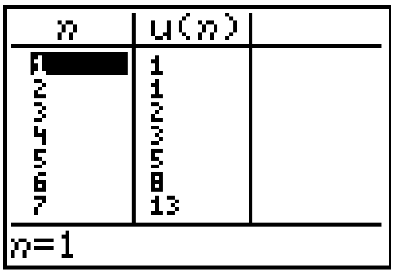
The Fibonacci sequence is defined, in calculator terms, by

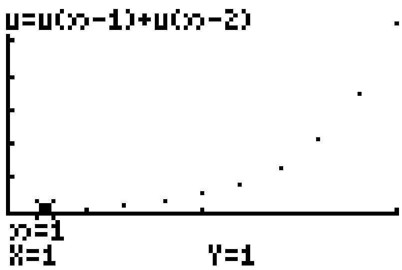
$$\mathrm{u}(n) = \mathrm{u}(n-1) + \mathrm{u}(n-2) \quad \text{with} \quad \mathrm{u}(1) = 1 \quad \mathrm{u}(2) = 1.$$

Press [Y=] and enter this formula (below left).

The two starting values u(1) and u(2) are stored in u(nMin) as a list $\{1, 1\}$.







The graph here is of the first 10 points of the sequence (nMax = 10) plotted with Xmin = 0, Xmax = 10, Xscl = 5, Ymin = −15, Ymax = 60 and Yscl = 10.

2.2 Using LIST commands

The *seq* command in the LIST (2nd STAT) OPS menu generates a sequence (list) specified by an explicit general term. The syntax is

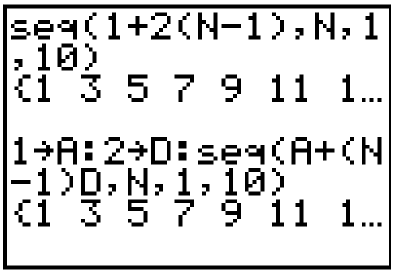
seq(general term, variable, start, end [, step]),

where variable can be any letter and *step* is assumed 1 if not given.[3] The total number of terms calculated must be less than 1000.[4]

2.2.1 Arithmetic progression

In the following sections, there is an implicit ENTER after each command.

seq(1 + 2(N−1), N, 1, 10)



The dots mean you can scroll across the answer using the arrow keys.

A nice way to use the general expression for an AP is

1 → A : 2 → D : seq(A+(N−1)D, N, 1, 10)   : is ALPHA .

Press ENTRY (2nd ENTER),[5] change the values of A and D, and press ENTER to re-execute the command.

2.2.2 Storing to a list

For analysis and graphing, it is often convenient to store a sequence in a named list, which is then stored in memory. The TI-83 has six built-in lists: L_1 – L_6; on the keyboard 2nd 1 – 2nd 6. Lists can also be given any four-character name.

To store a sequence to list L_1,

seq(2N−1, N, 1, 10) → L_1,

where → is the STO key. If you have already generated the sequence, just press → L_1.

Now press STAT 1 (Edit…) to enter the list editor. You can scroll up and down, graph, edit and carry out other operations on the list here.

[3] The variable in a *seq* command doesn't have to take integer values, for example *seq(sin(X), X, 0, 1, 0.05)* generates a sequence of values of the sine function.

[4] A list on the TI-83 can have a maximum of 999 entries. The same restriction applies to the *seq* command even if you are not storing the sequence in a list, because the TI-83 keeps the sequence in an internal list *Ans*.

[5] Pressing ENTRY once recalls the previous command, twice the command before that, and so on. There is limited memory for this, so you can't go back indefinitely.

2.2.3 Geometric progression

Try the following commands (ENTER after each).

seq(0.5∧(N−1), N, 1, 10)

MATH 1 a useful command for common sequences containing fractions

1 → A : 0.5 → R : seq(AR∧(N−1), N, 1, 10)

Exercise: Use ENTRY (2nd ENTER) to produce terms 11 to 20 in the sequence.

2.2.4 A neat trick for sequences

What we'd really like is an endless display of the sequence terms. This is provided in TABLE by evaluating the continuous function corresponding to our sequence at integer values of its argument. This is a good way to start sequences — see the Class Activity below.

Select *Func* in MODE to return to the normal type of graph.

In Y= : $Y_1 = 0.5 \wedge (X-1)$.

In TBLSET : TblStart = 1 ΔTbl = 1.

Press TABLE . Scroll down in the Y_1 column (the sequence terms) to see more digits at the bottom.

Provided TblStart and ΔTbl are integers, we get sequence values in the table. This is an alternative to using the sequence grapher to generate a table, and is usually faster in calculating the terms.

Class Activity

Give students one calculator between two. Have them press TBLSET (2nd WINDOW) and set both TblStart and ΔTbl to 1.

On the viewscreen calculator, with the OHP turned off, set $Y_1 = 2X + 1$ and press TABLE . Turn the OHP on and ask the students: *What's the rule?*

When they have worked it out as a class, press Y= to show them how the rule is entered. Have them enter the rule and generate the table. Write on the board that the rules to follow are all of the form __X + __.

Now, again with the OHP off, enter a different rule and show them the table. Ask them to make the table on their calculators the same.

Give them various rules to find, moving eventually to negative numbers for the coefficients. Ask them to summarise their findings regarding the two numbers in the rules.

3 Series

A series is the sum of the terms in a sequence, that is a sequence with + signs between successive terms. However, we also say *the sum of a series* to distinguish the sequence with + signs from the actual value when we carry out the additions. To calculate the sum of a series, we need to find the terms in the corresponding sequence, then add them up.

If a series has a finite number of terms, we just add them up to give the sum. All of what we do below can be applied to finite series. However, the more interesting series are infinite — we can't calculate the sum for these by carrying out the additions because there is an infinite number of them.[6] However, we can work out the sum of a finite number of terms, called a partial sum — the nth partial sum is the sum of the first n terms of the series. The behaviour of the partial sums as n gets bigger tells us something about the convergence of the series — whether the sum may be a finite number or infinite. The sum of an infinite series is defined as the limit of its partial sums as $n \to \infty$.

Both the sequence grapher and the LIST commands can be used to sum a series, but, unless we are quite clever (see Section 3.3 below), both methods require an explicit nth term. Because of this, the LIST commands are probably simpler, and the 'neat trick' above, suitably modified (see Section 3.2.3), allows us to use TABLE. However, the sequence grapher is easier to use if we want a graph of the partial sums.

The SERIES program (Section 3.3) combines the speed of the LIST commands with the flexibility of the sequence grapher. It is available at the website given on page 2.

3.1 Using the sequence grapher

The nth partial sum of a series S_n is the sum of the previous $n-1$ terms, S_{n-1}, plus the nth term; partial sums can therefore be defined recursively. In symbols,

$$S_n = S_{n-1} + u_n \qquad S_1 = u_1.$$

We'd like to generate the sequence of partial sums

$$S_2, S_3, S_4, S_5, \ldots.$$

We'll use the sequence v in the sequence grapher for the partial sums: v(n) will be the nth partial sum of some sequence whose nth term we have to specify.

Re-select Seq *in MODE to return to the sequence grapher if necessary.*

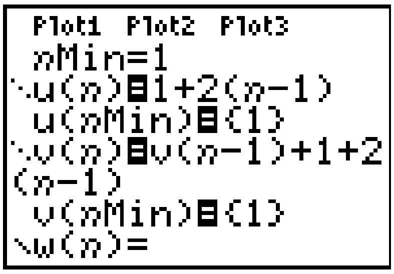
[6] There are alternative methods for summing some infinite series, one of the triumphs of Calculus.

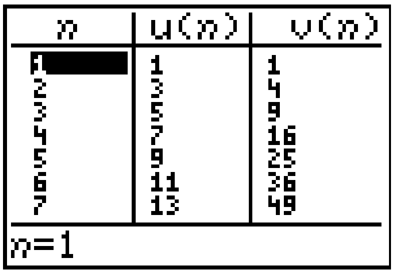
3.1.1 Arithmetic progression

For our AP, with nth term $\mathrm{u}(n) = 1 + 2(n-1)$, we have for the nth partial sum

$$\mathrm{v}(n) = \mathrm{v}(n-1) + 1 + 2(n-1) \quad \text{with} \quad \mathrm{v}(1) = 1,$$

as shown below left. The AP sequence is defined in $\mathrm{u}(n)$, but only so we can display both the sequence and the partial sums in the table — it's not necessary for calculating the partial sums.[7]





The table of partial sums can now be displayed as usual (above right). Here, the terms of the sequence are shown in the $\mathrm{u}(n)$ column and the corresponding partial sums in the $\mathrm{v}(n)$ column.

3.1.2 Geometric progression

Exercise: Set up the table of partial sums for our GP with $\mathrm{u}(n) = 0.5^{n-1}$. To what value does the (infinite) series appear to converge. Confirm your answer algebraically.

3.2 Using LIST commands

The *sum* command in the LIST MATH menu sums a sequence (list). The syntax is

sum(*list*),

where *list* can be a *seq* command.

The *cumSum* command in the LIST OPS menu generates a sequence of partial sums. The syntax is

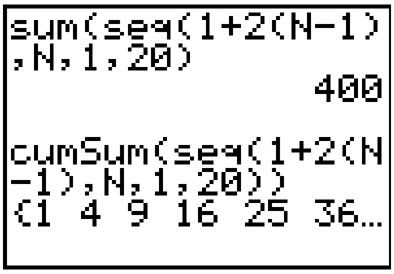
cumSum(*list*).

[7] We'd really like to calculate the nth partial sum by writing $\mathrm{v}(n) = \mathrm{v}(n-1) + \mathrm{u}(n)$, but the sequence grapher only lets us use the $(n-1)$th and $(n-2)$th terms in the definition of an nth term.

3.2.1 Arithmetic progression

sum(seq(1 + 2(N−1), N, 1, 20)) finds the sum of the first 20 terms in our AP.

cumSum(seq(1 + 2(N−1), N, 1, 20)) generates the first 20 partial sums of our AP: the ith entry is the sum of the first i terms.



The sum of an infinite series is defined as the limit as $n \to \infty$ of the nth partial sum. Scrolling across a list like the one above can give an idea of what that limit might be. Here, of course, there is no limit — the nth partial sum $\to \infty$ as $n \to \infty$.

If you prefer scrolling up and down, rather than side to side, store the partial sums to a list (STO L1 ENTER) and use STAT Edit.

3.2.2 Geometric progression

sum(seq(0.5∧(N−1), N, 1, 20))

1 → A : 0.5 → R : sum(seq(AR∧(N−1), N, 1, 20))

cumSum(seq(0.5∧(N−1), N, 1, 20))

Exercise: What's the sum of the GP $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$?

You might want to use ENTRY to change the end value of N in the last command above to be (almost) sure.

3.2.3 A neat trick for partial sums

Again we'd really like an endless sequence of partial sums. Try this.

Select *Func* in MODE.

In Y= : Y1 = sum(seq(0.5∧(N−1), N, 1, X)).

In TBLSET : TblStart=1 ΔTbl=1.

Press TABLE . Scroll down in the Y1 column (the partial sums) to see more digits at the bottom. *What's your best estimate for the sum?*

Why does this work? In generating the table, the calculator takes a given value of X, an integer in this case generated by the values of TblStart and ΔTbl, and puts it in the formula for Y1, giving the partial sum of X terms. It then displays the result in the Y1 column of the table. TblStart and ΔTbl must be integers.

3.3 Using the SERIES program

We can get around the problem of not being able to sum recursively defined sequences, such as the Fibonacci sequence, by a combination of methods. If we define the sequence as usual in u(n) and then use the 'neat-trick' idea by setting Y_2 = sum(seq(u(n), n, 1, X)), we end up with the Y_2 column in the table giving the partial sums of the sequence/series. If we also set Y_1 = u(X), we have the sequence values in the Y_1 column.

The SERIES program sums series defined either recursively (using the above method) or explicitly. Run the program,[8] select how the nth term is specified, enter the appropriate definition and

- generate a table of sequence values and corresponding partial sums. Once a table has been generated, you can change TblStart to look at the partial sums starting at different values of N, and increase ΔTbl to move through the partial sums more quickly, for example to examine the convergence or otherwise of the series.
- graph the partial sums. Somtimes a graph gives a better idea of where the partial sums are heading than a table.
- use FAST SUM to calculate a particular partial sum faster.

The SERIES program is a useful tool for investigating infinite series.

Exercise: Use the SERIES program to generate a table and a plot of the first twenty partial sums of

1. the GP with nth term $u(n) = 0.5^{n-1}$,
2. the Fibonacci sequence.

[8]Press PRGM and select the SERIES program by pressing the number against the program name or by scrolling down and pressing ENTER. Press ENTER again to run the program.

4 Questions on Sequences and Series

1. Generate a sequence of the cubes of first ten positive integers using a LIST command. Store the sequence in a list. Use the *sum* command to evaluate $1^3 + 2^3 + \cdots + 10^3$.

2. Find the sum of the first ten terms of the series $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots$.

3. (a) Use a single command to find the sum of the first 100 positive integers. Store the answer in memory S for use in (c).

(b) Edit your command in (a) (ENTRY) to find the sum of the *cubes* of the first 100 positive integers.

(c) Which is bigger: the sum of the cubes of the first 100 positive integers or the cube of the sum of the first 100 positive integers?

4. The half-life of a certain radioactive substance is one week. This means that, of the amount present at a particular time, only half will be left a week later. Suppose 1000 grams of the substance exists today, the beginning of Week 1.

(a) Write down the amount left at the beginning of Week 2, Week 3, ..., Week 10.

(b) Determine an infinite geometric sequence (recursive or explicit) that is a model of the amount of the substance at the beginning of Week n, where $n = 1, 2, 3, \ldots$. What is the common ratio of this sequence?

(c) When will there be only 0.005 grams remaining?

(d) How much of the substance was there a week ago (beginning of Week 0)?

(e) When will the substance be reduced to nothing *according to this model*?

5. The height of a particular fast-growing plant increases at the rate of 2.5% per month. Assume the plant is 30 cm high today and that it dies after 12 months.

(a) Determine a finite geometric sequence that is a model of the height of the plant after n months. Write out all the terms of the sequence. What is the common ratio?

(b) How long would the plant have to live to double in height??

6. Sue had \$1250 in a savings account three years ago. What will be the value of her account two years from now, assuming no deposits or withdrawals are made and the account earns 6.5% interest compounded annually?

7. Frank has \$12,876 in a savings account today. He made no deposits or withdrawals during the last six years. What was the value of his account six years ago? Assume that the account earned 5.75% interest compounded monthly.

8. Generating sequences recursively is equivalent to another mathematical process called iteration, in which we do the same operation over and over. Try some of the following sequences/iterations.

Generate some terms in each sequence. What happens to u(n) in each sequence as n becomes large? Be careful with brackets.

(a) $\mathrm{u}(n) = (\mathrm{u}(n-1))^2$. Try u(1) greater than 1; u(1) = 1 and −1; u(1) between −1 and 1; u(1) less than −1.

(b) $\mathrm{u}(n) = (\mathrm{u}(n-1))^2 - 1$ with values of u(1) between −2 and 2. Note that $\frac{1}{2}\left(1+\sqrt{5}\right) = 1.618\ldots$ and $\frac{1}{2}\left(1-\sqrt{5}\right) = 0.618\ldots$. What's special about these values for u(1)?

(c) $\mathrm{u}(n) = \sqrt{\mathrm{u}(n-1)}$.

(d) $\mathrm{u}(n) = \cos(\mathrm{u}(n-1))$.

(e) $\mathrm{u}(n) = \tan(\mathrm{u}(n-1))$.

See the reference at the end of Section 5.2 for more on iteration.

9. What value does the sequence $\left(1+\dfrac{1}{n}\right)^n$ approach as n gets larger and larger?

Hint: Use a sequence command with a step of at least 1000. Or apply the neat-trick method for sequences: set Y1 = $(1+1/\mathrm{X})^{\mathrm{X}}$ and use a suitably large ΔTbl, say 1000.

10. What is $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$? factorial ! is in the MATH PRB menu.

The *cumSum*(*seq* combination works well here, as we don't need too many terms to see the convergence of the partial sums. Store twenty or so terms in a list to make it easier to see the partial sums or use the neat trick for series. Note that the sum starts at $n = 0$.

What is $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}$? $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, where x is any number?

The following questions are from Chapter 9 of *Intermediate Algebra: Functions and Graphs* by K. Yoshiwara and B. Yoshiwara, Thomson Brooks/Cole, 2004.

11. Generate a table of values for each of the following recursive sequences. What happens to the terms as n gets larger? Do you recognise the number?

Hint: The answers for the first two sequences are square roots of particular numbers. *Can you make and test a conjecture here? Write down a sequence whose limit is 2;* $\sqrt{5}$.

The answers for the last two sequences should be obvious. *Can you make and test a conjecture here? Write down a sequence whose limit is 4; 5.*

(a) $a_1 = 1 \quad a_n = \dfrac{1}{1+a_{n-1}} + 1$ (b) $b_1 = 1 \quad b_n = \dfrac{2}{1+b_{n-1}} + 1$

(c) $s_1 = 1 \quad s_n = \dfrac{1}{2}\left(s_{n-1} + \dfrac{4}{s_{n-1}}\right)$ (d) $t_1 = 1 \quad t_n = \dfrac{1}{2}\left(t_{n-1} + \dfrac{9}{t_{n-1}}\right)$

12. A rubber ball is dropped from a height of 8 metres and returns to three-quarters of its previous height on each bounce.

(a) How high does the ball bounce after hitting the floor for the third time? for the tenth time?

(b) How far has the ball travelled vertically when it hits the floor for the fourth time? for the twentieth time?

13. According to legend, a man who had pleased the Persian king asked for the following reward. The man was to receive a single grain of wheat for the first square of the chessboard, two grains for the second square, four grains for the third square, and so on, doubling the amount for each square up to the 64th square. How many grains would he receive in all. (Fortunately the king had a good sense of humour.)

14. Find the sum of all integral multiples of 6 between

(a) 10 and 100.

(b) between 1 and 10,000.

5 Further Investigations

5.1 The Fibonacci sequence and the Golden Ratio

Here is one illustration of the many interesting properties of the Fibonacci sequence. For more, see the website *www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html*.

1. Generate a table of values of the sequence (be very careful with brackets)

$$u_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Do you recognise this sequence?

2. One of the many interesting things about the Fibonacci sequence becomes apparent when we look at ratios of successive terms. Using the sequence grapher, set

$$\mathrm{u}(n) = \mathrm{u}(n-1) + \mathrm{u}(n-2) \qquad \mathrm{u}(n\mathrm{Min}) = \{1, 1\}$$

to generate the Fibonacci sequence. Set

$$\mathrm{v}(n) = \mathrm{u}(n-1)/\mathrm{u}(n-2)$$

to calculate the ratios of successive terms.

Put TblStart = 3, ΔTbl = 1 and generate the table of values.

Scroll down and look at the v(n) column containing the ratios of successive terms of u(n). Does this sequence appear to be approaching a particular value? What value? Find this value accurate to six decimal places.

Now find a value for the Golden Ratio or Golden Section, $\frac{1+\sqrt{5}}{2}$.

What conclusion do you reach?

5.2 The logistic sequence

The logistic sequence or logistic map has become famous because it is one of the simplest sequences that exhibits chaotic behaviour. It also turns up in a number of areas such as population modelling.[9]

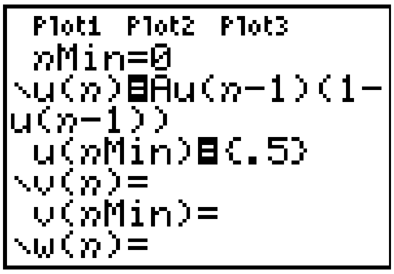
The logistic sequence is defined by (be very careful with brackets)

$$\mathrm{u}(n) = \mathrm{A}\mathrm{u}(n{-}1)\big(1{-}\mathrm{u}(n{-}1)\big),$$

where A is a constant.

The following figures show the set up for *time plots*, u(n) vs n, on the left and for *cobweb plots*, u(n) vs u($n-1$), on the right.

- Note that nMin = 0 here and we have taken u(0) = 0.5. The value for u(0) can be changed in subsequent plots.
- Store a value for A $\big(0 < \mathrm{A} \le 4\big)$ in memory A and press TRACE.
- For a cobweb plot, keep pressing the right arrow to generate the plot.[10]
  2nd QUIT returns you to the home screen.
- The TI-83 program CHAOS makes the process of graphing the logistic sequence a bit easier, allowing you to concentrate on the effects of changing A and u(0).

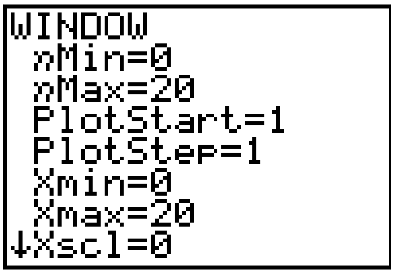


FORMAT

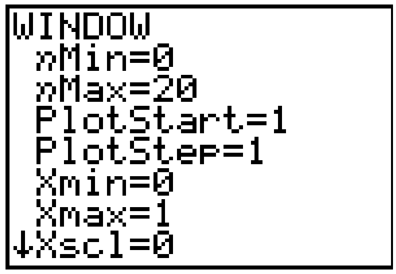
[9] For examples, see Sections 2.6, 2.7 of *Population Modelling* at *www.ma.adfa.edu.au* under *High School and College Activities*.

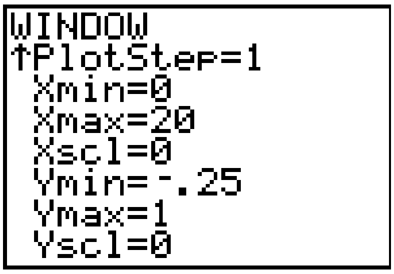
[10] In a cobweb plot, the curves $y = Ax(1-x)$ and $y = x$ are also plotted. The cobweb lines move between these two curves.

TIME **WEB**

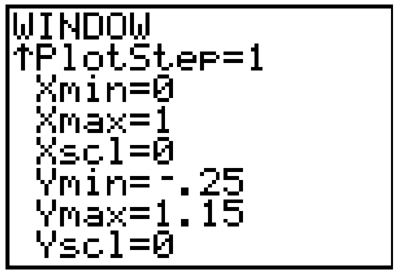


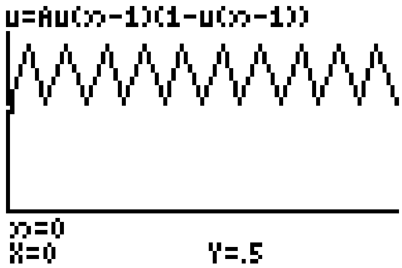
WINDOW



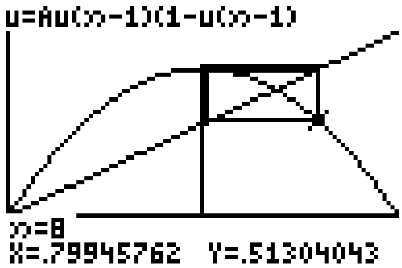


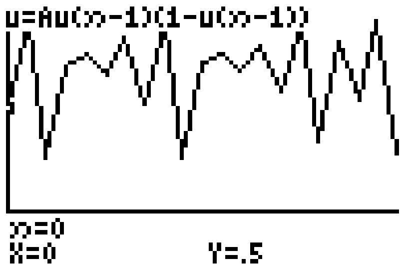
WINDOW



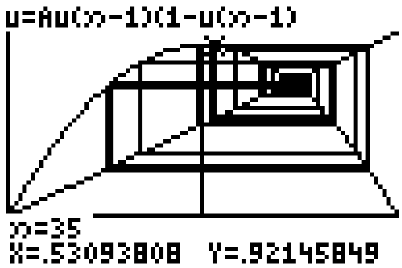


A = 3.2





A = 3.7



Exercise: Confirm graphically (by choosing appropriate values for A) that the sequence converges to 0 for $0 < A < 1$; to $1 - 1/A$ for $1 < A < 3$; and oscillates between 2 values, then 4 values, 8 values and so on as A is increased above 3. It becomes chaotic for values of A greater than about 3.568. Try both time and web plots.

For values of A greater than 4 or values of u(0) less than 0 or greater than 1, the sequence diverges rapidly to $\pm\infty$.

The book *Mathematics by Computer: Iteration* by Lynne Kelly, Wizard Books, 1996 has some good worksheets and discussion on iteration in general, including a section on *The Logistic Equation*.

5.3 The harmonic series

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

has many interesting properties.[11]

1. Use the 'neat-trick' method to generate the partial sums, starting with the first.

   Then press [TBLSET] and change *Indpnt* to *Ask* using the cursor and [ENTER]. Press [TABLE]. Now you can input the number of terms (X) in the partial sum and the calculator will calculate the corresponding partial sum (Y). *Does the series appear to be converging?*

2. The *seq* command can only generate up to 999 terms. To sum more terms than this, we need a program. Use FAST SUM in the SERIES program (Section 3.3) to find the sum of the harmonic series to 1000, 2000 and 4000 terms. Record the number of terms and the partial sum in a table, continuing on from your calculator table in 1. Continue the table for as long as you have patience. Doing something else while the program runs is a good idea. *Does the series appear to be converging?*

3. Graph the data from your table in 2 using STATPLOT (see below). *What can you conclude about the harmonic series?* The SERIES program could also be used.

4. There is an approximation to the partial sums of the harmonic series,

$$\sum_{n=1}^{m} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{m} \approx \frac{1}{2m} + \ln(m) + \gamma,$$

   where $\gamma = 0.5772\dots$ is the Euler or Euler-Mascheroni constant.

   Check this approximation against the values you found in 1 and 2. To compare with your values in 1, set $Y_2 = (2X)^{-1} + \ln(X) + 0.5772$ and use the TABLE. What do you conclude?

5. Find the next few decimal places in the Euler-Mascheroni constant.

6. Find out what you can about the Euler-Mascheroni constant.

Graphing data

Enter your data into lists L_1 (for X) and L_2 (for Y) using [STAT] Edit. . . .

Press [STAT PLOT] ([2nd] [Y=]) to set up the graph defined by the list values. With the arrow keys, cursor and [ENTER], select respectively *Plot1*; *On*; *Scatter Plot* (the first); *L1*; *L2*; *square marker.*

[11] The series does not converge — the more more terms you take, the larger the sum, despite the fact that the terms tend to 0 as $n \to \infty$.

Press Y= and turn off any functions (cursor on = sign and press ENTER). Note that Plot1 is highlighted, meaning it will be plotted when you press GRAPH or TRACE. You can turn plots on and off here in the same way you do functions.

Now set a WINDOW manually as you would for a function or press ZOOM 9 (ZoomStat) for an automatic version. Press TRACE to trace along the points.

6 Solutions to Questions

1. The command is seq(X^3, X, 1, 10) [STO] L1. You can use any letter instead of X: X is easiest to use because it requires only one key press. If you just execute the *seq* command, the combination [STO] L1 on the next line produces the output Ans → L1 and stores the sequence.

To calculate the sum, use sum(L1) to produce the answer 3025.

2. The command is sum(seq(1/X^3, X, 1, 10)), giving 1.197531986.

3. (a) The sum of the first 100 positive integers is sum(seq(X, X, 1, 100) = 5050.

(b) The sum of the *cubes* of the first 100 positive integers is sum(seq(X^3, X, 1, 100) = 25,502,500.

(c) The cube of the sum of the first 100 positive integers is $5050^3 \approx 1.29 \times 10^{11}$, larger than the sum of the cubes of the first 100 positive integers, which is only about 2.55×10^7.

4. (a) The amount of radioactive substance in grams at the beginning of successive weeks, starting at Week 1, is (to two decimal places)

$$1000,\ 500,\ 250,\ 125,\ 62.5,\ 31.25,\ 15.63,\ 7.81,\ 3.91,\ 1.95.$$

(b) Explicitly, the amount at the beginning of Week n is $u_n = 1000\big(0.5\big)^{n-1}$. Recursively, $\mathrm{u}(n) = 0.5\mathrm{u}(n-1)$, with $\mathrm{u}(1) = 1000$. The common ratio here is 0.5.

(c) Using a calculator table with $\mathrm{Y}_1 = 1000\big(0.5\big)^{\mathrm{X}-1}$ or a table generated from the recursive definition using the sequence grapher, there is 0.005 g remaining sometime in Week 18, i.e. between $n = 18$ and $n = 19$.

(d) A week ago (beginning of Week 0), there was twice as much as there is now (beginning of Week 1), that is 2000 g.

(e) According to this model, there will always be some of the substance left, although the amount becomes small very rapidly. You can't reduce any number to 0 by dividing it by 2 or raising it to a power.

5. (a) The height n months from now is $u_n = 30\left(1 + \frac{2.5}{100}\right)^n = 30\big(1.025\big)^n$, $n = 1, 2, \ldots, 12$. The table below shows months (top row) and corresponding heights, rounded to one decimal place.

0	1	2	3	4	5	6	7	8	9	10	11	12
30	30.8	31.5	32.3	33.1	33.9	34.8	35.7	36.6	37.5	38.4	39.4	40.4

The common ratio is 1.025.

(b) Using the calculator table (remember to use X instead of n if you don't use the sequence grapher), the plant would double in height in the 29th month.

6. Let S_n be the amount of money in Sue's account at the start of year n, with $n = 1$ corresponding to 3 years ago. Then,

$$S_n = 1250\left(1 + \frac{6.5}{100}\right)^{n-1} = 1250\big(1.065\big)^{n-1}.$$

In two year's time, $n = 1 + 5 = 6$, so the amount of money in her account will be

$$S_5 = 1250\big(1.065\big)^5 = \$1712.61 \quad \text{to the nearest cent.}$$

You could also use the calculator table to reach this answer.

7. The amount in Frank's account is given by

$$S_n = 12876\left(1 + \frac{5.75}{12 \times 100}\right)^{12(n-1)} = 12876\left(1 + \frac{0.0575}{12}\right)^{12(n-1)},$$

where $n = 1$ corresponds to now.

Six years ago, $n = -5$, and the amount in Frank's account was

$$S_{-5} = 12876\left(1 + \frac{0.0575}{12}\right)^{-72} = \$9126.56 \quad \text{to the nearest cent.}$$

You could also use the calculator table (scroll up) to reach this answer.

8. (a) Set $\mathrm{u}(n) = \big(\mathrm{u}(n-1)\big)^2$ in the sequence grapher, set $\mathrm{u}\big(n\mathrm{Min}\big)$ to an appropriate starting value and use the table to see what happens as n becomes large.

Initial value	Limit of sequence
$\mathrm{u}(1) > 1$	∞
$\mathrm{u}(1) = 1$	1
$-1 < \mathrm{u}(1) < 1$	0
$\mathrm{u}(1) = -1$	1
$\mathrm{u}(1) < -1$	∞

(b) Set $\mathrm{u}(n) = \big(\mathrm{u}(n-1)\big)^2 - 1$.

Initial value	Limit of sequence
$\mathrm{u}(1) < -1.618\ldots$	∞
$\mathrm{u}(1) = -1.618\ldots$	$-1.618\ldots$ (constant)
$-1.618\ldots < \mathrm{u}(1) < -0.618\ldots$	oscillates between 0 and -1
$\mathrm{u}(1) = -0.618\ldots$	$-0.618\ldots$ (constant)
$-0.618\ldots < \mathrm{u}(1) < 0.618\ldots$	oscillates between 0 and -1
$\mathrm{u}(1) = 0.618\ldots$	$-0.618\ldots$ (constant)
$0.618\ldots < \mathrm{u}(1) < 1.618\ldots$	oscillates between 0 and -1
$\mathrm{u}(1) = 1.618\ldots$	$1.618\ldots$ (constant)
$\mathrm{u}(1) > 1.618\ldots$	∞

(c) $\mathrm{u}(n) = \sqrt{\mathrm{u}(n-1)}$ gives 0 if $\mathrm{u}(1) = 0$, and tends to 1 otherwise.

(d) $\mathrm{u}(n) = \cos\big(\mathrm{u}(n-1)\big)$ tends to 0.7391... in radian MODE and 0.99985... in degree MODE.

(e) $\mathrm{u}(n) = \tan\big(\mathrm{u}(n-1)\big)$ looks several times as though it is going to settle down to a limit, but never does.

9. The sequence $\left(1 + \dfrac{1}{n}\right)^n$ approaches the value $e = 2.71828\ldots$ as n goes to ∞.

10. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e \qquad \sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2 \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$

11. (a) approaches $\sqrt{2}$; (b) approaches $\sqrt{3}$; (c) approaches 2; (d) approaches 3.

12. (a) The following table gives the rebound height after bounce n.

n	1	2	3	4	...
Height (m)	6	$\frac{9}{2}$	$\frac{27}{8}$	$\frac{81}{32}$	...

The rebound height after the third bounce is therefore 27/8 m.

The rebound height after bounce n is $6\big(0.75\big)^{n-1}$ m. Therefore, after the tenth bounce, the rebound height is $6\big(0.75\big)^9 \approx 0.45$ m. Alternatively, put $\mathrm{Y}_1 = 6\big(0.75\big)^{\mathrm{X}-1}$ and use the table.

(b) The total distance travelled is 8 + 2 × each rebound height, i.e.

$$8 + 2\left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \ldots\right).$$

When the ball hits the floor for the fourth time, it has travelled

$$8 + 2\left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32}\right) = 35.75\,\text{m}.$$

The distance travelled when it hits the floor for the bth time ($b \geq 2$) is therefore

$$8 + 2\sum_{n=1}^{b-1} 6\big(0.75\big)^{n-1} = 8 + 12\sum_{n=1}^{b-1} 0.75^{n-1}.$$

Therefore, when the ball hits the floor for the twentieth time, it has travelled

$$8 + 12\sum_{n=1}^{19} 0.75^{n-1} \approx 55.8\,\text{m}.$$

Here, use the command 8 + 12 sum(seq(0.75∧(B−1), B, 1, 19)) to do the calculation.

13. Number of grains = 1 + 2 + 4 + 8 + 16 + ... = $\sum_{n=1}^{64} 2^{n-1} \approx 1.89 \times 10^{19}$.

Use sum(seq(2∧(N−1), N, 1, 64)) to calculate this.

To put this number into context, 10^{18} grains would be about the same volume as the Great Wall of China and the five Great Pyramids combined.[12]

14. We need to evaluate $\sum_{n=a}^{b} 6n$, where

(a) $6a$ is the smallest multiple of 6 greater than 10 and $6b$ is the largest multiple of 6 less than 100. Clearly, $a = 2$ and $b = 16$, so that the required sum is

$$\sum_{n=2}^{16} 6n = 810,$$

where we use the command sum(seq(6N, N, 2, 16)).

(b) $6a$ is the smallest multiple of 6 greater than 1 and $6b$ is the largest multiple of 6 less than 10,000. Clearly, $a = 1$ and b is the integer part of 10,000/6, that is 1666. The required sum is

$$\sum_{n=1}^{1666} 6n = 8,331,666.$$

We can't use the single command sum(seq(6N, N, 1, 1666)) here because there are more than 999 terms in the series. Either break the sum up into two parts, sum(seq(6N, N, 1, 999)) and sum(seq(6N, N, 1000, 1666)) and add the two answers, or use the SERIES program.

[12] From *Large Numbers* by Victor Scharaschkin, Australian Senior Mathematics Journal 4 (2), 111–125 (1990).