Calculus on a CFX-9850GB

Peter McIntyre

Contents

ADFA **CMA**

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At *www.unsw.adfa.edu.au/pems/news/high_school/hsc_activities.html*

- A variety of graphics-calculator activities for Years 9 and 10 — written as part of the CQTP Program for Sharp, Casio and TI calculators.
- *Using the CFX-9850GB* — an introduction to the basic operations, suitable for Years 8–12.
- *The Graphics Screen and Accuracy* — information to help you understand the graphical and numerical limitations of a graphics calculator.
- *Coordinate Geometry on a CFX-9850GB* — basic commands and a variety of problems, suitable for Years 9 and 10.
- *Population Modelling* — a variety of problems from simple exponential growth to Leslie matrices and difference equations, covering Years 7–12.
- *Sequences and Series on an CFX-9850GB* — basic commands and a variety of problems, suitable for Years 10–12.
- *Matrices on an CFX-9850GB* — suitable for Years 11 and 12.
- *Complex Numbers on a CFX-9850* — suitable for Year 12.
- *Introduction to Complex Numbers* — complex numbers from the beginning, covering the basic operations, but set in the context of complex numbers as a mathematical structure.

All the programs listed in these notes can be found at the above web site. You will need a PC-Link Kit FA-123 to copy these programs from your computer to your calculator.

1 Introduction

Graphics calculators lend themselves very nicely to demonstrating the visual aspects of Calculus — graphs of functions, tangent lines, areas under curves, etc — as well as to calculating numerically (approximating) many of the quantities that arise — derivatives and definite integrals, maximum and minimum values, etc.[1]

They can be used at a number of levels.

- As a basic graph plotter — what does the graph of $y = e^x$ look like?
- To investigate 'what if' questions, for example what happens if you change the parameters a and b in the equation $y = ae^{bx}$?
- To do (numerically) many of the basic calculations in Calculus, such as finding the slope at a point on a graph, definite integrals, maxima and minima, etc.
- To illustrate graphically, perhaps by way of a program, some of the concepts of Calculus. Two examples are showing how a secant line tends to a tangent line in the appropriate limit and how we can approximate the area under a graph by the areas of some rectangles. With sufficient ingenuity, almost anything can be done here, the only limitation being the small screen of the calculator.
- To automate, using the built-in operations or programs, some of the calculations that arise in Calculus: numerical integration methods, solving differential equations numerically and so on.

At a more mundane level, graphics calculators are fun. Students pick up the operations very quickly (much faster than teachers), and if you can't get your students to use a graphics calculator, there are heaps of games to tempt them.

The other good news is that there are lots of resources available, many free on the Web, as well as a rapidly increasing number of books on using graphics calculators in almost every aspect of Mathematics and Science.

Getting started is always the hardest, especially when you have to modify or write new courses, but the experience at ADFA and most other schools and universities at which graphics calculators have been used for a while, is that graphics calculators should not just be an add-on to a course, but should be integrated fully, including their use in tests and exams. This raises some issues, most of which are resolvable. You might like to read for example *Graphics calculators in the mathematics curriculum: Integration or differentiation?* by Jen Bradley, Barry Kissane and Marian Kemp about their experiences in WA.[2]

At UNSW@ADFA, we have been using TI calculators (TI-83+s at present) in our first-year courses since 1993 and have come to appreciate their worth in learning Mathematics.

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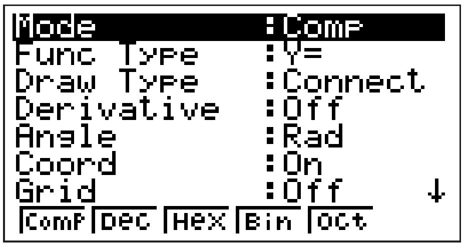
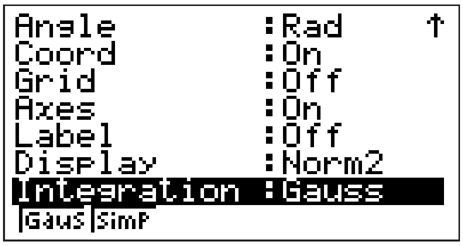
[1] There are, of course, 'calculators' that go even further and do things symbolically.

[2] This paper is available at *wwwstaff.murdoch.edu.au/∼kissane* under Publications. There are a number of other interesting papers here too.

2 Basic Calculus Operations

Before starting, press [MENU] [1] to go to the RUN screen, then [SET UP] ([SHIFT] [BS]).
Set your calculator as shown and press [EXIT].

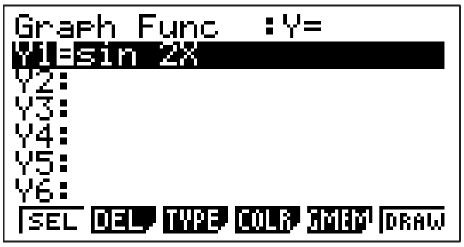




1. Graph $f(x) = \sin(2x)$ for $0 < x < \pi$

- Press [MENU] [5]: set Y1 = sin 2X.
  The independent variable X is the [X, θ, T] key in the fourth row of keys.

  Note the highlighted = sign, which means the function will be plotted when you press [DRAW]. Use [F1] (SEL) to toggle the function off/on.

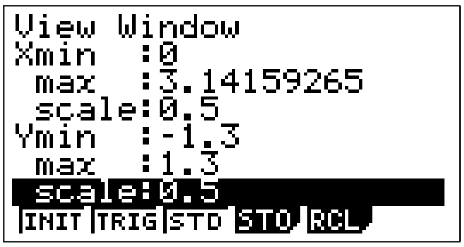


- Press [SHIFT] [F3] (V-Window): specify the viewing window.
  - For X, suitable values here are:
    Xmin = 0   *Xmax = π*   *Xscale = 0.5.*

    Press [EXE] or the down arrow to move between values, *including the last.*

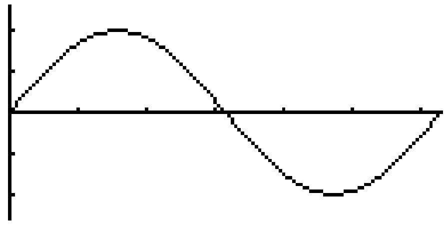
    π is [SHIFT] [EXP].

    Xscale is the distance between tick marks on the X axis (cosmetic only: 0 gives no tick marks).

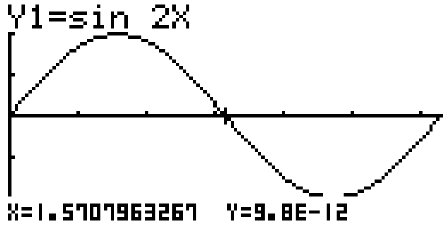


  - Suitable Y values are: *Ymin* = −1.3
    Ymax = 1.3   *Yscale = 0.5.*
  - Note the difference between the subtract key [−] and the change-sign key [(−)].
  - Press [EXIT] to return to the *Graph Func* screen.

- Press F5 (DRAW): graph the function.

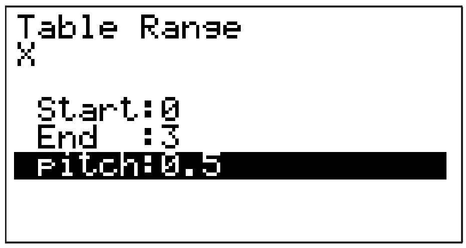


- Press F1 (TRACE): move the cursor along the graph with the left/right arrows; the coordinates of the point on the graph are shown at the bottom.

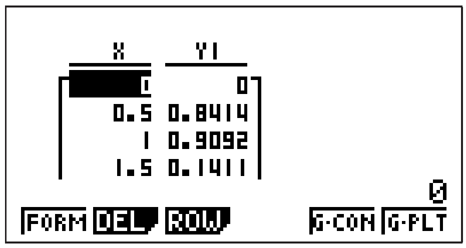


- **TABLE**
  - Press MENU 7 to select TABLE mode.
  - Set the table 'WINDOW' using F5 (RANG):
    Set *Start = 0, End = 3, Pitch = 0.5*, pressing EXE after each.

    Press EXIT to return to *Table Func*.



- Press F6 (TABL). Use the arrow keys to move around the table.

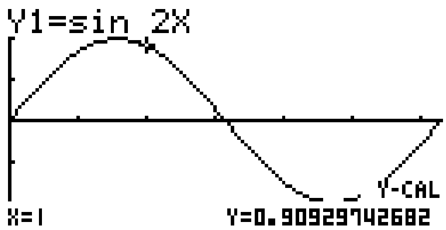


2. Estimate $f(1)$

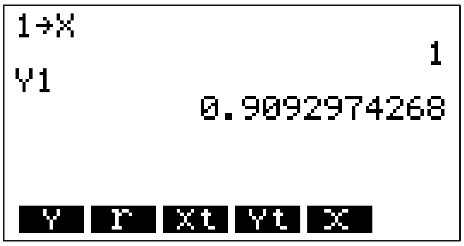
- **On the GRAPH screen**
  - Press MENU 5 F6 to redraw the graph.
  - Press F5 (G-Solv) F6 (▷) F1 (Y·CAL).

    Type the X value, 1 EXE, to move to the desired point on the graph. Note the coordinates at the bottom of the screen.

    Alternatively, use the left/right arrows to move the cursor along the curve in TRACE (but note the problem that arises when trying to reach X=1). The up and down arrows move between functions if there is more than one graphed.



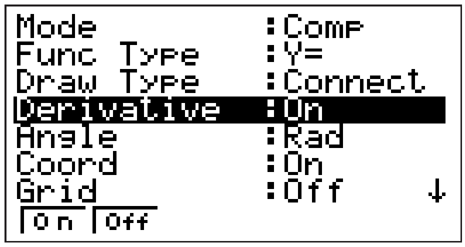
- **On the RUN screen** [MENU] [1]
  - Store 1 in X: [1] [→] [X, θ, T] [EXE].
    Type in [Y] [1] [EXE].
    Y is [VARS] [F4] [F1].
    You can't just type [Y].

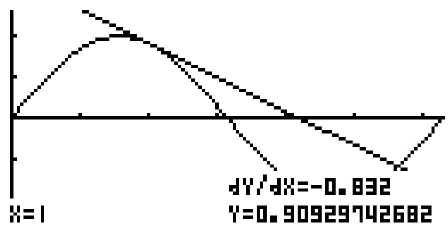


- *Answer*: $f(1) = 0.90930$, rounded to 5 decimal places.

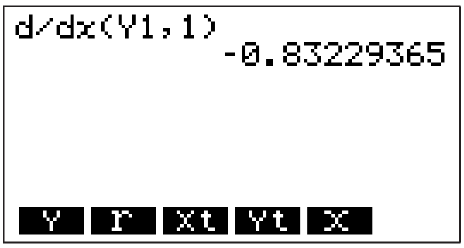
3. **Estimate $f'(1)$**

- **On the GRAPH screen**
  - Press [SET UP] ([SHIFT] [MENU]) and set *Derivative On*. Press [EXIT].
  - Press [MENU] [5] [F6] to redraw the graph.
  - Press [TRACE] and use the arrow keys to move the cursor along the curve; the value of the derivative is shown at each point. Unfortunately, it is not possible to go to a particular point — even *Y·CAL* in the G-Solv menu only gives the function value, and not the derivative, at the specified point.
  - The only way to see a value on the graph seems to be by using *Tangent* (TANG) in the [SKETCH] menu from the RUN screen: *Tangent Y1, 1* [EXE] draws the tangent at X =1 and displays the derivative.





- **On the RUN screen**
  - *d/dx (Y1, 1)*.
    d/dx: [OPTN] [F4] (CALC) [F2].
    Y: [VARS] [F4] (GRPH) [F1].



- *Answer*: $f'(1) = -0.83229$, rounded to 5 decimal places.
- *Accuracy*: can be adjusted in *d/dx* by an optional third argument.

4. Estimate $\int_0^{\pi} \sin(2x)\,dx$

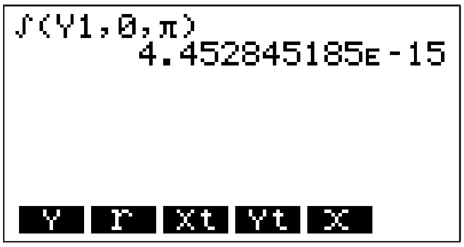
- **On the GRAPH screen**
  - Press [MENU] [5] [F6] to redraw the graph.
  - Select $\int dx$: [G-Solv] [F6] [F3].

    Move the cursor respectively to the lower and upper integration limits when prompted, followed each time by [EXE].

- **On the RUN screen**
  - $\int(Y1,\,0,\,\pi)$.

    $\int dx$: [OPTN] [F4] [F4].

    Y: [VARS] [F4] (GRPH) [F1].

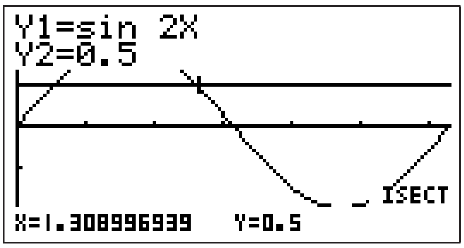


- *Answer*: $\int_0^{\pi} \sin(2x)\,dx = 0.00000$, rounded to 5 decimal places. The E$^-$15 stands for *times* 10^{-15}.
- *Accuracy*: can be adjusted in $\int dx$ by an optional fourth argument.

5. Solve $\sin(2x) = 0.5$ for $0 \le x \le \pi$

- **On the GRAPH screen**
  - Graph Y1 and Y2 = 0.5.
  - Use *ISCT*, [F5] in the [G-Solv] menu. The calculator will find the first intersection; press the right arrow to find further intersections.

    Note that, because the right-hand side of the equation here is a constant, we could have used *X·CAL* in the [G-Solv] menu. The intersection method here can be used no matter what the right-hand side.

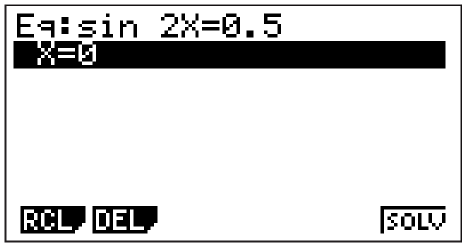


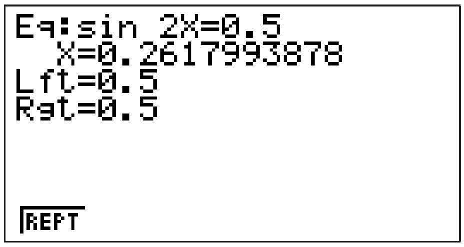
- **On the EQUATION screen**
  - Use *Equation Solver*: [MENU] [A] [F3].

    Enter the equation as

    $$\sin 2\mathrm{X} = 0.5$$

    and press [EXE]. Enter a guess for X (this will determine which of the two possible values you find), press [EXE] and [F6] (SOLV). Press [F1] to repeat the calculation with a different guess.

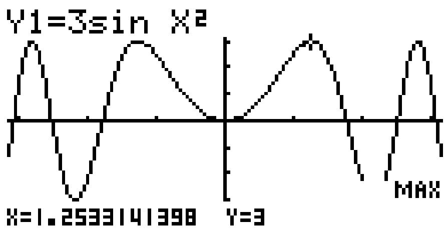




- *Answer*: the curves intersect at $x = 0.26180$ (Guess: 0) and $x = 1.30900$ (Guess: 2), both rounded to 5 decimal places.

6. Find the value of the first positive maximum of $3\sin(x^2)$

- **On the GRAPH screen**
  - Press [MENU] [5] and set Y1 = $3\sin \mathrm{X}^2$. Turn off or delete all other functions. Set a [V-Window] of $[-\pi, \pi, 1] \times [-4, 4, 1]$ and plot the function.
  - Select *MAX* in the [G-Solv] menu. The calculator will find the first (local) maximum from the left on the screen. Press the right arrow to find successive maxima until you reach the one you want.
  - Press [2nd] [QUIT] and type $2\mathrm{X}^2$ [EXE] on the home screen. You should recognise the first 6 digits of the number.

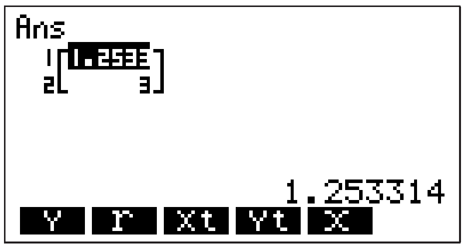


- **On the RUN screen**
  - *FMax (Y1, 1, 1.5).*

    FMax: [OPTN] [F4] (CALC) [F6] [F2].

    Y: [VARS] [F4] (GRPH) [F1].

    The last two inputs are the bounds for the search.



- *Answer*: the maximum value of 3 occurs at $x = 1.25331$, rounded to 5 decimal places.

The [G-Solv] operations *ROOT*, *MIN* and *Y·ICPT* work in the same way as *MAX*.

3 A Classic Problem

A hare and tortoise compete in a one-kilometre race. The distance each competitor has travelled from the starting point is given by a formula. In time t **minutes**, the distance in **metres** travelled by the hare is given by $H(t) = \frac{500}{3}(2\sqrt{t} + \sqrt[3]{t})$, while the distance in **metres** travelled by the tortoise is given by $T(t) = 100t + 250\sqrt{t}$.

Press MENU 5 (GRAPH) and enter the formulas for H and T in Y1 and Y2 respectively. You have to use X (X, θ,T) as the independent variable. The cube root is SHIFT (.

Set your *View Window* (SHIFT F3) so that the two graphs will go from the bottom left to the top right of the screen. *Hints*: The race takes about 5 minutes. How far is the race?

If you select *Simul Graph On* in the SET UP menu of your calculator before graphing, you will get a real-time view of the race.

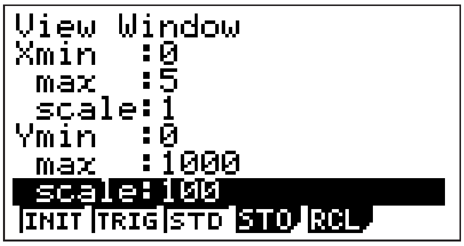
Answer the following questions, writing down the steps you took. Trace, *ISCT* (intersection), *Y·CAL* (find y given x) and *X·CAL* (find x given y) in the G-Solv menu will be helpful. Press the right arrow after any of these operations to find further values.

1. Who gets to the halfway point first? How long does it takes them? Verify your answer algebraically.
2. What is the time and distance at which the two runners are neck and neck?
3. Who wins the race, by what time margin and by what distance margin?

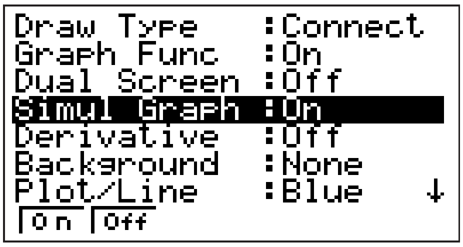
Teacher's Notes

The questions in this version have been written in general terms deliberately for a good class. For a less-advanced class, students may need to be led a little through each question. For example: *What equation do we need to solve to answer this question? What does this mean about the graphs of each side of the equation? How do we solve this equation on the calculator?* and so on.

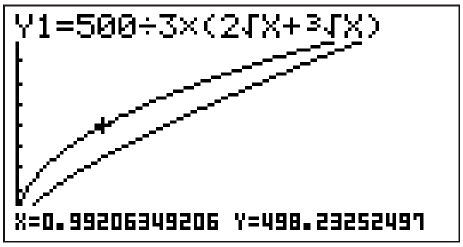
Press [MENU] [5] (GRAPH) and put the equation for the hare in Y1 and that for the tortoise in Y2. You might like to discuss with the class how to write the formulas in a suitable form for the calculator. Time t becomes X on the calculator.



Then set the *View Window*: discuss first with the class what each axis represents and suitable scales. The Y axis is distance in metres, so 0 < Y < 1000. The winner is then the competitor whose graph first reaches the top of the screen (assuming *Simul Graph On*).

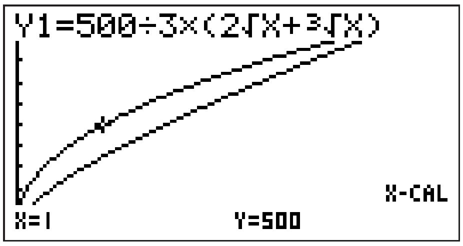


The time (X) scale has to be guessed. The race takes a little under 5 minutes, so 0 < X < 5 gives a good view.
Press [EXIT] to return to the *Graph Func* screen.



1. Press [F6] (DRAW), then [F1] (Trace): use the up/down arrows to see which graph is which.

   The hare clearly reaches the halfway point (500 m) first.

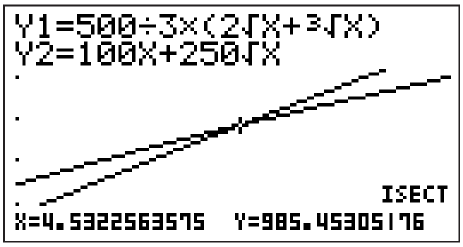


   To find how long the hare took, solve $H(t) = 500$ for t using *X·CAL* ([G-Solv] [F6] [F2]). Select the appropriate curve (Y1) using the up/down arrow keys and press [EXE].
   Type in the Y value (500) and press [EXE] again.

   The value for t is 1 minute, a value we can confirm algebraically to be exact by substituting $t = 1$ into the equation for the hare. Note that it is easy to *verify* that $t = 1$ is a solution, but tricky to *solve* $H(t) = 500$ algebraically (it turns into a cubic equation).

   The hare reaches the halfway point first in a time of 1 minute.

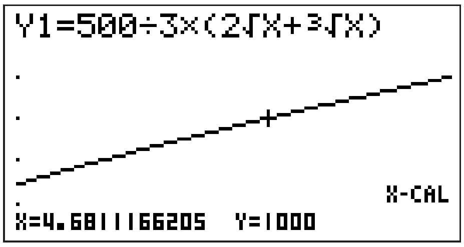
2. To find when they are neck and neck, we have to solve $H(t) = T(t)$, that is find the intersection (*ISCT*) of Y1 and Y2 (algebraically, this turns into a quartic equation). We obtain $t = 4.53$ minutes and a distance of 985 m, both accurate to 3 significant digits. It might be useful to *Zoom IN* (F2 F3) on this part of the graph (as we did for the figure) to see the two curves more clearly.

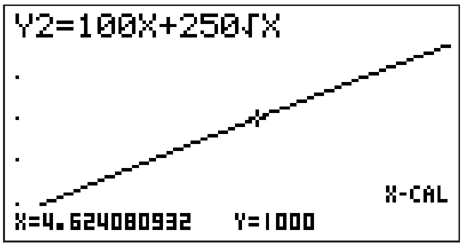


The hare and tortoise are neck and neck after about 4.53 minutes or about 4 minutes 32 seconds, 985 metres from the start.

3. To find the winner, we have to determine the time at which each competitor reaches the finish (1000 m).

Choosing the appropriate curve with the arrow keys (or graphing one curve at a time) and using *X·CAL*, we find the hare finishes at $t = 4.681$ minutes and the tortoise finishes at $t = 4.624$ minutes.





To find the distance margin, calculate $H(4.624)$, the position of the hare when the tortoise finishes, using *Y·CAL*: $H(4.624) = 994.45$ m, rounded to 5 significant digits.

The tortoise wins the race by a margin of 0.057 minutes or 3.42 seconds. The distance margin is 5.55 m.

Using the Newton-Raphson Method

The problems here are also a good application of the Newton-Raphson method for finding the zero of a function. For this, it is useful to have a program.

The program NEWTON for the CFX-9850 can be downloaded at *www.ma.adfa.edu.au* under *High School and College Activities*. The program finds the zero of a graphed function, given an initial guess provided by the position of the cursor on the graph using Trace .

The program finds zeroes of the function in Y1, so it is useful to put the hare and tortoise equations in Y2 and Y3. Y1 can then be defined as Y1 = Y2 − 500 to find when the hare reaches halfway; Y1 = Y2 − Y3 to find when they are neck and neck; Y1 = Y2 − 1000 to find when the hare finishes and Y1 = Y3 − 1000 to find when the tortoise finishes.

4 Maximum and Minimum

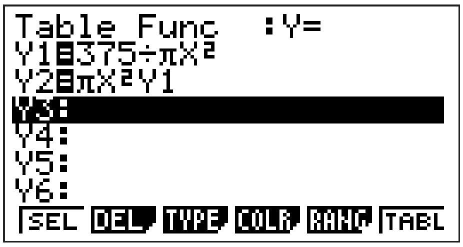
When manufacturers are designing their packaging, they must keep in mind the amount of product that has to fit inside and the amount of material it will take to make the package. Consider the humble soft-drink can. The standard volume is 375 mL or 375 cm^3. Any number of cans can be designed that will hold this volume of liquid, but they will vary in shape and therefore in the amount of material needed to make the can (and therefore cost).

The formula for the volume of a cylinder V in terms of radius r and height h, is

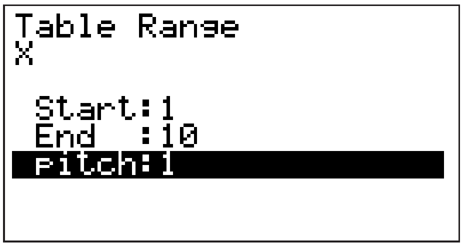
$$V = \pi r^2 h.$$

Rearrange the volume formula to make h the subject and let the volume be 375 cm^3. *What are the units of r and h?*

$h =$



Press MENU 7 and enter this formula as Y1, with X as radius r.
As a check, enter the volume formula:
Y2 = π X^2 Y1.
Y1 is VARS F4 (GRPH) F1.



Press F5 (RANG) : set *Start* = 1, *End* = 10 and *Pitch* = 1.
Press EXIT, then F6 (TABL).
Do you get the correct value for the volume in Y2?

Write down the formula for the surface area of a cylinder, including the ends.
The surface area determines the amount of material needed to make the can. *Why?* Press F1 (FORM) and enter the formula for surface area in Y2 in terms of X (radius) and Y1 (height).

SA =

Y2 =

View the table of values again and scroll down. *What do you notice about the values of the surface area?*

Use RANG (F1 F5) to set new starting values and smaller increments to find the minimum surface area and corresponding radius (radius accurate to 1 decimal place).

Minimum SA =

Radius =

Now press [MENU] [5], graph the surface area as a function of radius and use *MIN* in the [G-Solv] menu to find the minimum.

Draw your graph here.

Write down your values for the radius, height, surface area and circumference of the can when the surface area is a minimum. *Do these values seem reasonable?*

How does this compare with a can of soft drink? Why the differences? What about other cans?

You might like to read the article *The Best Shape for a Tin Can* by P. L. Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993).

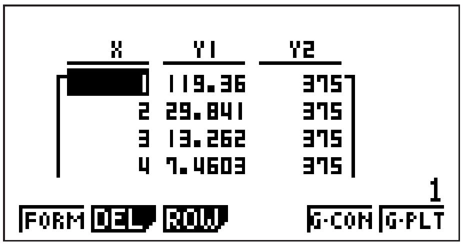
Teacher's Notes

This is an activity from: *Integrating the Graphics Calculator into Year 9 and Year 10 of the Victorian Mathematics CSF*, Teachers Teaching with Technology (T^3), 1998, modified somewhat. There are a number of maximum/minimum activities for all years that work in the same way.

The height of the cylinder is given by $Y1 = 375/\pi X^2$, where X is the radius r.

As a check, enter the volume formula $Y2 = \pi X^2 Y1$.

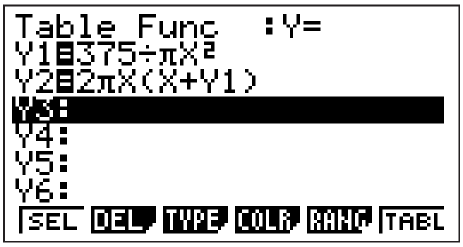
Press [F6] (TABL).



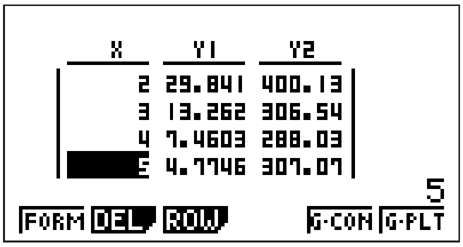
The total volume of the metal used to make the can, assuming the walls are of uniform thickness, is just the surface area times the thickness. Minimum surface area therefore means minimum volume of metal.

The surface area of a cylinder, including the ends, is given by

$$SA = 2\pi r^2 + 2\pi r h = 2\pi r(r + h).$$

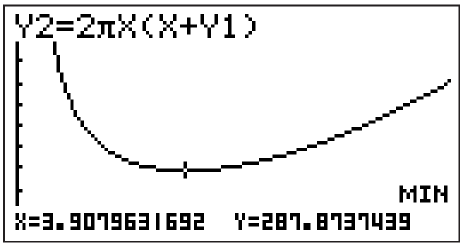


View the table of values again and scroll down. *What do you notice about the values of the surface area?*
The surface area decreases then increases as the radius increases. There is a (local) minimum.



With *Start* = 3, *End* = 4 and *Pitch* = 0.1, we find a radius of 3.9 cm for minimum surface area.

Alternatively, one can view the graph of surface area Y2 versus radius X, and use *MIN* in the [G-Solv] menu to find the minimum. The figure has a [WINDOW] of $[0, 10, 2] \times [0, 1000, 100]$.



Again, we obtain a value of $r = 3.9$ cm (accurate to 1 decimal place) for the radius giving minimum surface area.

If your students have sufficiently developed calculus skills, they could prove algebraically that the global minimum lies at $r = \sqrt[3]{375/2\pi} \approx 3.9$. More generally, for a given volume V, it is not too hard to show that $h = 2r$ (height = diameter) for the least surface area.

Is this a reasonable answer for the radius?

Other factors may have to be taken into account such as what circumference is comfortable for the average human hand, the wastage of material when cutting the ends and the cost of making the joins.

radius $r = \sqrt[3]{\dfrac{375}{2\pi}} \approx 3.9\,\text{cm}$ height $h = \sqrt[3]{\dfrac{1500}{\pi}} \approx 7.8\,\text{cm}$ ratio $\dfrac{h}{r} = 2$

surface area $\approx 288\,\text{cm}^2$ circumference $\approx 24.6\,\text{cm}$

How does this compare with a can of soft drink?

A soft-drink can has a radius of 3.25 cm and a height of 13 cm: $h/r = 4$. Its surface area is about $332\,\text{cm}^2$ and circumference 20.4 cm. The article *The Best Shape for a Tin Can*[3] by P. L. Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993) goes into why there might be differences between the theory here and the actual values. A good example of mathematical modelling.

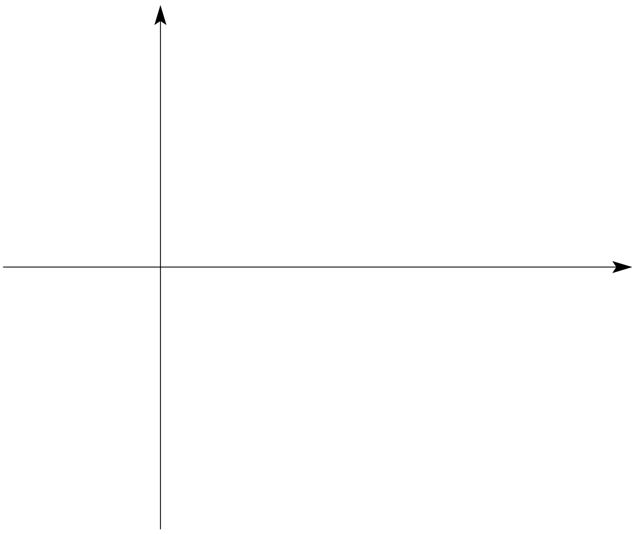
[3] A copy of this article is available from Peter McIntyre.

5 Derivatives and the Shape of a Graph

From: *This is I.T. Graphics-calculator activities for upper-secondary students* by Pat Forster, Alan Cadby and Gary Young, AAMT, 1998.

This investigation is to help you understand how the first and second derivatives of an equation can tell you the shape of its graph. You will also learn some new terminology to describe graphs.

1. Copy from your calculator the graph of $y = 5x^2 - 2x^5$ for $-1 \leq x \leq 2$, $-10 < y < 10$. Put scales on your graph.



2. Find and enter the equations for the first and second derivatives of $y = 5x^2 - 2x^5$ into your calculator as Y2 and Y3. Look at the values of these derivatives for different values of x using the calculator's table of values as follows.

Press MENU 7, then F5 (RANG) and set *Start* = -1; *End* = 2 and *Pitch* = 0.2.

Then press EXIT and F6 (TABL).

(a) Use a red pen and put plus signs (+) along the sections of your graph above where dy/dx is positive. Similarly, put minus signs ($-$) and zero (0), where appropriate.

(b) Using a different-coloured pen, mark where d^2y/dx^2 is +, $-$ or zero. (Scroll up the table in the X column.)

(c) The points on a graph where $dy/dx = 0$ are called **stationary points**. Fill in the table below.

Type of stationary point	Co-ordinates	$\frac{dy}{dx}$ $(+, -, 0)$	$\frac{d^2y}{dx^2}$ $(+, -, 0)$
maximum			
minimum			

3. **(a)** Sections of a graph where $\frac{d^2y}{dx^2}$ is positive are described as **concave up**.

If $\frac{d^2y}{dx^2}$ is negative the curve is **concave down**.

For what values of x is your graph concave up? ....

For what values of x is your graph concave down? ....

Is the graph at the maximum point concave up or concave down? ....

Is the graph at the minimum point concave up or concave down? ....

(b) A point at which a curve is concave down (d^2y/dx^2 negative) on one side and concave up (d^2y/dx^2 positive) on the other side, *or vice versa*, is called a **point of inflection**. The sign of the second derivative must *change* as we pass through a point of inflection because the graph changes concavity.

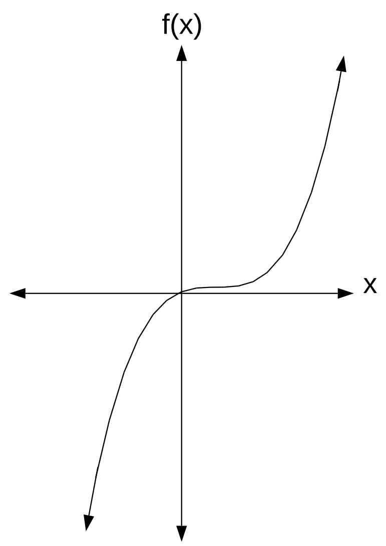
Use the table of values to find the point of inflection of the function here: change *Start*, *End* and *Pitch* to zoom in on the zero of Y3. *Why do we look for a zero? Is this zero a point of inflection?* Write its co-ordinates on your graph.

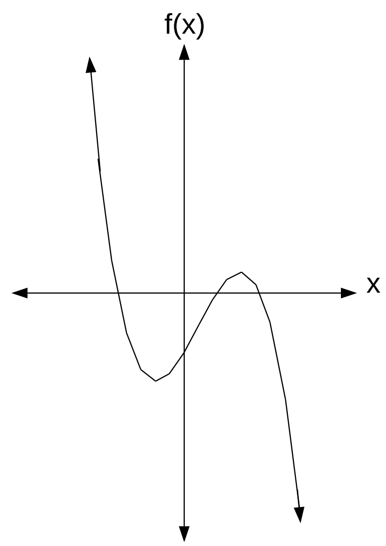
(c) *Is $x = 0$ a point of inflection of the function $y = x^4$?* Give reasons.

(d) Points of inflection can be the steepest section of a graph — look at the graph of $y = \sqrt[3]{x-1}$ at $x = 1$. They can also be stationary points (graph is horizontal) — look at the graph of $y = x^3 + 1$ at $x = 0$. However, in general, the slope at a point of inflection can be any value.

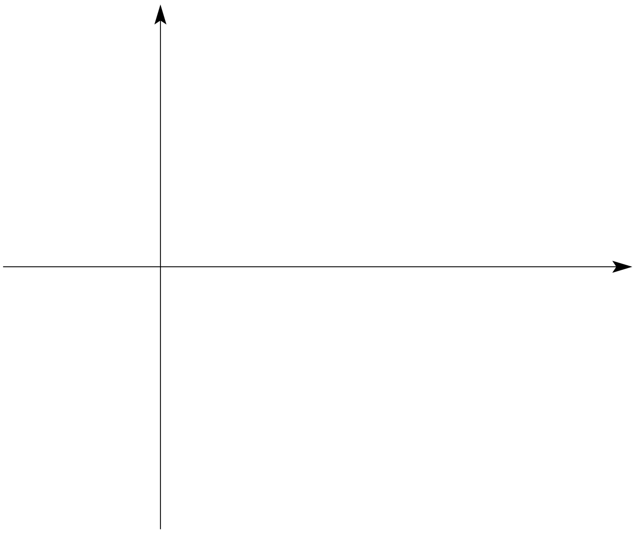
Put a cross on the points of inflection in each of the following two graphs. *Is the point of inflection in either graph a stationary point?*

....





4. Sketch a graph with the given properties.



(a) Endpoints $(-2, 1)$ and (7,6).

(b) Stationary point at $(1, -5)$ where $\dfrac{d^2y}{dx^2}$ is positive.

(c) Stationary point at (3,2) where $\dfrac{d^2y}{dx^2} = 0$.

(d) $\dfrac{d^2y}{dx^2} = 0$ at $(2, -1)$, $\dfrac{d^2y}{dx^2}$ is negative between $x = 2$ and $x = 3$, and $\dfrac{d^2y}{dx^2}$ is positive for $x > 3$.

(e) Classify the points in (b) – (d) as a maximum, minimum or point of inflection.

....

(f) For what values of x is the curve concave up?

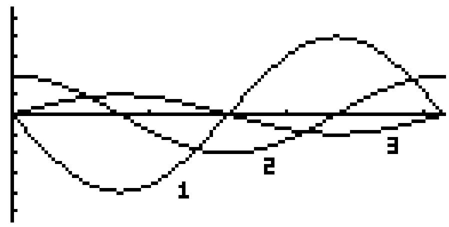
....

6 Graphing Derivatives

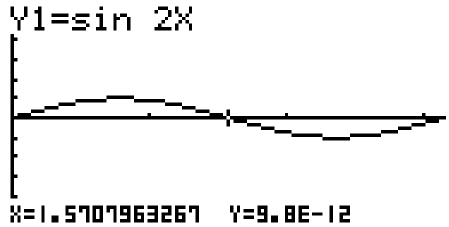
Based on: *How do you graph derivatives and anti-derivatives?* byJohn Maloney, Eightysomething! 7(1), 1997.

Here is a question to challenge your students' understanding of the concept of derivative.

The figure shows graphs of a function, its derivative and its second derivative. *Which curve is which?*



Let's investigate how to graph a function, and its first two derivatives. Press SET UP and make sure that the *Func Type* is *Y=* and that *Angle* is *Rad*.



Press MENU 5: set Y1 = sin 2X.

Set V-Window to $[0, \pi, 1] \times [-5.5, 5.5, 1]$.

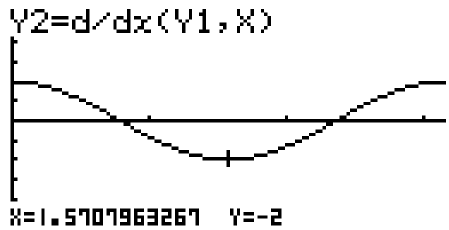
Press EXIT, then F6 (DRAW) and F1 (Trace).

Plotting derivatives

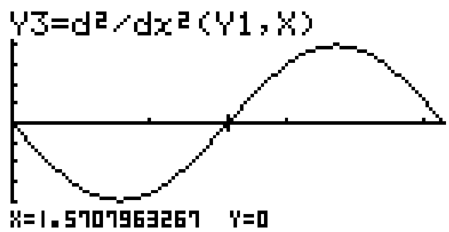
To graph the first derivative, set Y2 = d/dx (Y1, X). *d/dx* is OPTN F2 F1; *Y* is VARS F4 F1.

Y1 is the function we are differentiating and the X is the value at which we wish to calculate the derivative — this value is set by the grapher as it plots successive points on the graph.

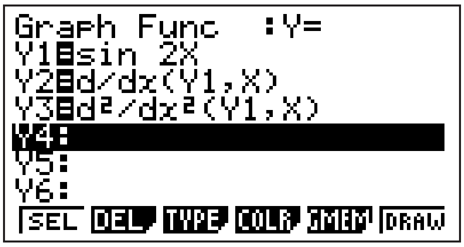
Pressing EXIT twice, EXE to store the function, then F6 F1 gives you the figure.



Set Y3 = d^2/dx^2 (Y1, X) for the second derivative of Y1. d^2/dx^2 is OPTN F2 (CALC) F2.



With Y1, Y2 and Y3 turned on, you should obtain the graph of the question above. Students, of course, must use the relationship between a function and its derivative to answer the question.



Answer to the Question

One possible answer

Curve 3 is initially positive, so that it cannot be the derivative of either curve 1 or curve 2, both of which have negative slopes initially. Therefore, curve 3 must be the function.

The initial slope of curve 3 is positive, so that curve 1, which is initially negative, cannot be the first derivative of curve 3. Therefore, curve 2, which is initially positive, must be the first derivative of curve 3.

Curve 1, initially negative, must be the derivative of curve 2, and therefore the second derivative of curve 3.

7 Rectangles, Area and the Definite Integral

The CFX-9850 program RIEMANN illustrates graphically how the area under a graph can be approximated by the areas of rectangles. As the number of rectangles covering the area increases, we obtain a better approximation to the area. Here we approximate $\int_0^1 e^x\,dx$ by drawing rectangles (the sum of the areas is a *Riemann sum*).

- Press [MENU] [5] and put the function $f(x) = e^x$ in Y1: Y1 = e(X).
- Set a [V-Window] of $[0, 1, 0.2] \times [-0.3, 3, 1]$, pressing [EXE] after each value. The value of -0.3 for *Ymin* allows room at the bottom of the screen for displaying the cursor coordinates. Press [EXIT].
- Press [MENU] [B], scroll down to RIEMANN and press [F1] to start the program.
- Set the integration limits A = 0 and B = 1.
- Set the number of rectangles $N = 5$.
- Choose the Left-Endpoint Rule (LER in the table below). The program will plot the function and draw in 5 rectangles, each rectangle touching the curve at its top-left corner. In this case, the area of the rectangles clearly underestimates the area under the graph.
- Press [EXE] to see the area of the rectangles as an approximation to the area under the graph.
- Press [EXE] and set $N = 5$ again, but this time choose the Right-Endpoint Rule (RER). Now we obtain an overestimate of the area under the curve.
- Repeat the above steps, doubling the number of rectangles each time. Fill in the table below.[4] Your best estimate: $\int_0^1 e^x\,dx \approx$ ____________.

N	LER	RER	MEAN
5			
10			
20			
40			
80			

- When you've had enough, press [AC/ON] twice to stop the program.

[4]The mean of the two estimates is equivalent to the Trapezoidal-Rule approximation to the area, a more accurate approximation for a given N than either the Left- or Right-Endpoint Rules.

Teacher's Notes

The Casio expression for e^x is eX, where the e is obtained by pressing [SHIFT] [ln]. I prefer to put in brackets, e(X), so it doesn't look like a multiplication.

Although the e^x key ([SHIFT] [ln]) puts e on the screen, it must have a number after it. To obtain the value of e, type e1 or e(1).

Answers *rounded to three decimal places.*

N	LER	RER	MEAN
5	1.552	1.896	1.724
10	1.664	1.806	1.720
20	1.676	1.762	1.719
40	1.697	1.740	1.718
80	1.708	1.729	1.718

Best estimate is 1.718.

The exact answer is $e - 1 = 1.718$ to three decimal places.

Note that if you type [SHIFT] [ln] [−] [1], showing e−1 on the screen, it gives you $1/e$!

8 Approximating Definite Integrals

Modified from an ADFA Lab, which is itself based on a lab in *Resources for Calculus, Volume 1: Learning by Discovery*, Anita Solow, editor, Mathematical Association of America Note 26, 1993.

In this lab, we shall be comparing several numerical approximations to

$$\int_0^1 (5x^4 - 3x^2 + 1)\,dx$$

with the exact answer obtained by algebraic integration. This will give us a feel for some of the methods of numerical integration, which we can then use for any function, including those which cannot be integrated algebraically.

Question 1 *Algebraic integration — the exact answer*

What is the exact value of this integral? You may not realise it, but you are using the *Fundamental Theorem of Calculus* to do this definite integral exactly.

Question 2 *Left-Endpoint and Right-Endpoint Rules*

One approach to numerical integration is to approximate the definite integral of $y = f(x)$ with $a \le x \le b$ by the areas of a number of rectangles under the curve. If a left-hand corner of each rectangle touches the curve, we have the *Left-Endpoint Rule*; if a right-hand corner of each rectangle touches the curve, we have the *Right-Endpoint Rule*. As the number of rectangles in the interval $[a, b]$ gets larger and larger (covering the integration range $a \le x \le b$ with more and more rectangles), both Rules give numbers closer and closer to the definite integral (exact answer).

(a) On Figure 1 (at the end of this Lab), sketch and shade in the rectangles for the Left-Endpoint-Rule approximation to the definite integral $\int_a^b f(x)\,dx$ with $n = 4$ (four rectangles). *Note*: The function in Figure 1 is not the function in Question 1.

(b) Using your sketch in (a), explain why the Left-Endpoint Rule with four rectangles approximates the area under the graph as

$$hf(x_0) + hf(x_1) + hf(x_2) + hf(x_3),$$

where $x_0 = a$, $x_4 = b$ and the width of each rectangle is $h = (b - a)/4$.

(c) Use the RIEMANN program (instructions over) to estimate $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using the Left-Endpoint Rule with $n = 4$. A suitable View Window is $[0, 1, 0.1] \times [-0.3, 3, 0.5]$. Note that the integrand here is positive, so that the definite integral corresponds to the area under the graph of f.

(d) Now use the RIEMANN program, doubling n, the number of rectangles, until two successive answers for the Left-Endpoint Rule are the same when rounded to two decimal places. Write down the n value of the first of these two answers.

Question 3 *The Trapezoidal Rule*

The Left-Endpoint and Right-Endpoint Rules approximate the area under a function by rectangles. In many cases, for example the function in Figure 1 with the rectangles you drew in, this is not a good approximation. We get a better approximation by using trapeziums: both top corners of each trapezium touch the curve.

(a) On Figure 2, draw in and shade the trapeziums, the total area of which approximates the definite integral $\int_a^b f(x)\,dx$, again with $n = 4$.

The area of the trapezium in Figure 3 is $h(r + s)/2$. To see this result, split the trapezium into two regions — a triangle and a rectangle.

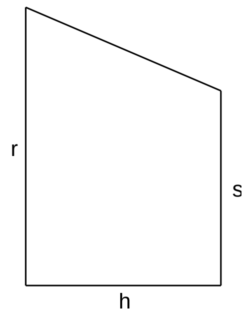


Figure 3

(b) Using your sketch in (a) and Figure 3, explain why the Trapezoidal Rule with four trapeziums approximates the area under the graph as

$$hf(x_0) + 2hf(x_1) + 2hf(x_2) + hf(x_3),$$

where $x_0 = a$, $x_4 = b$ and the width of each trapezium is $h = (b - a)/4$.

(c) Evaluate T_4, the Trapezoidal Rule with four trapeziums, as an estimate of the integral $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using the NUMINT program (T stands for Trapezoidal Rule). How does this result compare with the rectangle result and the exact answer?

(d) Now use the NUMINT program, doubling n, the number of trapeziums, until two successive answers for the Trapezoidal Rule are the same when rounded to two decimal places. Write down the n value of the first of these two answers. Compare it with the rectangle n value.

Question 4 *Simpson's Rule*

A picture of Simpson's Rule for which $n = 4$ is given in Figure 4. We want to estimate the area under the solid curve. We do this by fitting parabolas to three successive points on the graph and adding up the areas under the parabolas.

The dashed line in Figure 4 shows two parabolas: one through $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$; the other through $\big(x_2, f(x_2)\big)$, $\big(x_3, f(x_3)\big)$ and $\big(x_4, f(x_4)\big)$.

(a) On Figure 4, shade the area calculated by Simpson's Rule as an approximation to the definite integral $\displaystyle\int_a^b f(x)\,dx$.

(b) Evaluate S_4, Simpson's Rule with four divisions of the integration interval, as an estimate of $\displaystyle\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ (Simpson's Rule) using NUMINT. Note that the NUMINT version of Simpson's Rule uses 2N sub-divisions, where N is the value you input (n, the number of divisions of the integration interval, must be even for Simpson's Rule). Compare your result with those from Questions 1 – 3.

(c) Now use the NUMINT program, doubling n, the number of divisions of the integration interval, until two successive answers for Simpson's Rule are the same when rounded to two decimal places. Write down the n value of the first of these two answers. Compare it with the rectangle and trapezium values.

Question 5 *Comparing the methods*

What conclusions can you draw from your results regarding the different methods for estimating the definite integral? Which method would you choose to use? Why?

Programs

These programs calculate approximate values for $\displaystyle\int_A^B f(X)\;dX$. The number N is an input to the program.

NUMINT approximates the integral using the *Left-Endpoint Rule* (**L**), the *Right-Endpoint Rule* (**R**), the *Trapezoidal Rule* (**T**) and the *Midpoint Rule* (**M**), **all with N sub-divisions**, and *Simpson's Rule* (**S**) **with 2N sub-divisions** to ensure an even number of sub-divisions.

RIEMANN approximates the integral using the *Left-Endpoint Rule* or *Right-Endpoint Rule* with N sub-divisions of the interval [A, B], and draws the corresponding rectangles.

Use: Type the function to be integrated into Y1. [AC/ON] twice stops both programs.

- For NUMINT, run the program and follow the prompts. Press [EXE] repeatedly to obtain the respective answers, and finally to input a different number N of sub-divisions.
- For RIEMANN, first set a suitable WINDOW to display the function (plot the function on the integration range first to check). Run the program and follow the prompts. Make sure B > A, otherwise things get mixed up. Press [EXE] after the graph is plotted to see the numerical approximation to the integral, and [EXE] again to do a new plot.

Numerical Integration Lab Figures

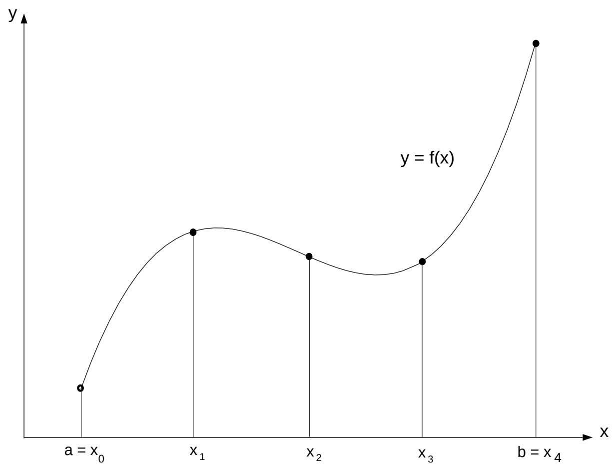


Figure 1: Left-Endpoint Rule

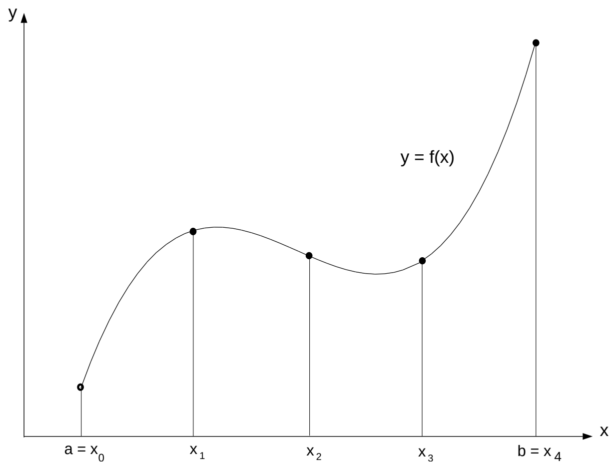


Figure 2: Trapezoidal Rule

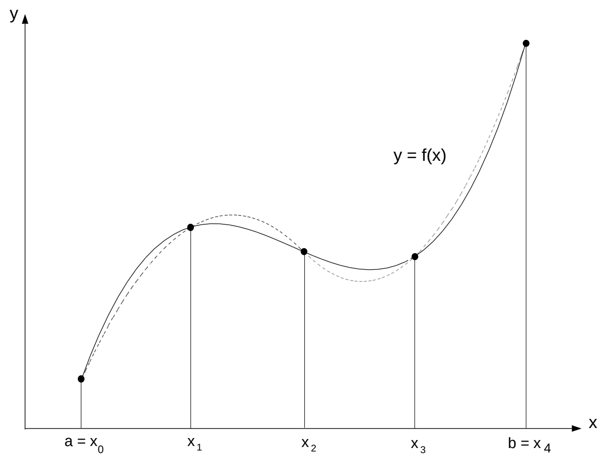


Figure 4: Simpson's Rule

9 Web Sites

- A good starting point for web links is the AAMT home page *www.aamt.edu.au*, in particular the links to other Maths sites under *Resources / Members' Sites / Mathematical Software and Technology*.

- Barry Kissane's home page *wwwstaff.murdoch.edu.au/∼kissane* is another web site of general interest. This page contains some interesting discussion papers under Publications and lots of other useful stuff, including resources.

- *www.prenhall.com/divisions/esm/app/calculator* contains instructions on how to do most things with most models of graphics calculator. A terrific reference site for basic calculator operations.

- *www.casio.edu.shriro.com.au* The local Casio site.

- *www.casio.co.jp/edu_e/index.html* The Japanese Casio site (in English).

- *www.charliewatson.com.au/casio* Charlie Watson's Australian Casio site.