Matrices on a CFX-9850GB

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At *www.unsw.adfa.edu.au/pems/news/high_school/hsc_activities.html*

- A variety of graphics-calculator activities for Years 9 and 10 — written as part of the CQTP Program for Sharp, Casio and TI calculators.
- *Using the CFX-9850GB* — an introduction to the basic operations, suitable for Years 8–12.
- *The Graphics Screen and Accuracy* — information to help you understand the graphical and numerical limitations of a graphics calculator.
- *Coordinate Geometry on a CFX-9850GB* — basic commands and a variety of problems, suitable for Years 9 and 10.
- *Population Modelling* — a variety of problems from simple exponential growth to Leslie matrices and difference equations, covering Years 7–12.
- *Sequences and Series on an CFX-9850GB* — basic commands and a variety of problems, suitable for Years 10–12.
- *Calculus on an CFX-9850GB* — suitable for Years 11 and 12.
- *Complex Numbers on a CFX-9850* — suitable for Year 12.
- *Introduction to Complex Numbers* — complex numbers from the beginning, covering the basic operations, but set in the context of complex numbers as a mathematical structure.

The programs used in these notes can be found at the above web site. You will need a PC-Link Kit FA-123 to copy the program from your computer to your calculator.

1 The Basics

The basic matrix operations work just like the operations with numbers.

Let's learn by working out the matrix product $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Call the first matrix $\boldsymbol{A}$, the second $\boldsymbol{B}$.

1.1 Putting matrices in your calculator

Press [MENU] [3] or move the cursor to the MAT icon and press [EXE]. You are now ready to edit one of the twenty-six matrices Mat A – Mat Z.

With the cursor on Mat A, first input the order (rows × columns), each followed by [EXE]. Then the elements: type in a value and [EXE]. The cursor moves on and leads you through the elements row by row. Make sure you press [EXE] after the last element.

Press [EXIT] and input Mat B.

Press [MENU] [1] to return to the RUN screen and [OPTN] [F2] to select the matrix menu.

1.2 Displaying matrices

You call up matrices by pressing [F1] (Mat) and the appropriate letter. For example, pressing [F1] [ALPHA] [B] will display 'Mat B' on the screen. Pressing [EXE] will show you Mat B.

From now on, we omit the [ALPHA] before letters.

1.3 Multiplying matrices

Press [F1] [A] [F1] [B] to display 'Mat A Mat B' and press [EXE] to display the numerical result

$$\boldsymbol{AB} = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}.$$

The result is stored in Mat Ans.

If you now want to multiply by $\boldsymbol{A}$ again, i.e. to find $\boldsymbol{A}(\boldsymbol{AB})$, press [F1] [A] [F1] [Ans] (on the [(–)] key) to display 'Mat A Mat Ans' and then [EXE] to give the numerical result

$$\boldsymbol{A}(\boldsymbol{AB}) = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

1.4 Squaring matrices

We could also work out $\boldsymbol{A}(\boldsymbol{AB})$ as $\boldsymbol{A^2B}$.

Press F1 A x^2 F1 B to display 'Mat A^2 Mat B' and EXE to give the same result

$$\boldsymbol{A}^2\boldsymbol{B} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Integer powers of a matrix are produced the same way as with numbers (for positive integers). Only square matrices can be raised to a power.

$$\boldsymbol{A}^4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$

The only negative integer 'power' that works is -1 (using the x^{-1} key): however, Mat A^{-1} produces the **inverse** of Mat A (see Section 3). The reciprocal of a matrix is not defined.

1.5 More involved expressions

Extensions work just as you would expect. For example, to work out $\boldsymbol{AB} + 3\boldsymbol{B}$, press F1 A F1 B + 3 F1 B to display 'Mat A Mat B + 3 Mat B' and EXE to give

$$\boldsymbol{AB} + 3\boldsymbol{B} = \begin{bmatrix} 7 & 11 & 15 \\ 14 & 19 & 24 \end{bmatrix}.$$

1.6 Storing matrices

If you wanted to keep the previous answer for later use, you might store it in $\boldsymbol{C}$ by pressing F1 Ans $\rightarrow$ F1 C EXE.

Notice that the calculator automatically makes $\boldsymbol{C}$ have the correct order.

1.7 Illegal operations

The calculator will not let you do invalid operations. For example, if you try to calculate $\boldsymbol{A} + \boldsymbol{B}$ by entering [F1] [A] + [F1] [B], you will see 'Mat A + Mat B' on the screen, but pressing [EXE] produces the message Dim ERROR. *Why?*

1.8 Other matrix operations

A number of operations are contained in the [OPTN] MAT menu.

- *determinant*: [F3] [1] [A] gives Det Mat A, with value -2.
- *transpose*: [F4] [B] gives Trn Mat B, with result $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.
- *dimension*: [F6] [F2] [F6] [F1] gives Dim Mat B, with result $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
- *'filling' a matrix*: [F6] [F3] [(] [1] [,] [F6] [F1] [A] [)] gives Fill(1, Mat A) and produces a matrix of '1's (press [F1] [A] [EXE] to see this) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
- *identity matrix*: [F6] [F1] n produces the $n \times n$ identity matrix.

See the CFX-9850 Guidebook for details on the other MAT menu items.

2 Gauss Elimination

2.1 The method

The endpoint of Gauss elimination or Gauss reduction is a matrix in **row-echelon** form, characterised by:

(a) the first non-zero element in each row is a 1 (called a **pivot**);

(b) all elements in the column below a pivot are 0.

To reduce a matrix to row-echelon form, we use **elementary row operations**.[1] The three elementary row operations are

(A) exchange two rows ($R_i \rightleftharpoons R_j$)

(B) multiply a row by a constant ($R_i \to cR_i,\ c \neq 0$)

(C) add a constant multiple of one row to another ($R_i \to R_i + cR_j,\ c \neq 0$).

Procedure

1. If the first column of the matrix is all zeros, “cross” out thfis column to leave a smaller matrix.

2. If the element in the first row and first column (top left element) of the matrix is 0, exchange Row 1 with another row (Operation A).

(Sometimes it is convenient to do this even if the top left element is non-zero, so as to avoid fractions in Step 3.)

3. Multiply the top row by a constant to make the first element 1 (Operation B).

4. Use Operation C to obtain ‘0’s **below** the 1 (pivot) by adding multiples of **Row 1** to each successive row ($R_i \to R_i + cR_1,\ i = 2, 3, \ldots$).

5. “Cross out” the first row and first column to leave a smaller matrix.

6. Go back and start at Step 1 on this smaller matrix.

[1] Each row operation corresponds to the multiplication of the matrix by a corresponding elementary matrix.

Notes

- The only choice in this version of Gauss elimination is which rows to interchange in Step 2. All the other operations are prescribed.
- The solutions to the simultaneous equations described by the row-echelon matrix are the same as those of the original matrix, hence the usefulness of the method. Write down the equations corresponding to the row-echelon matrix and use back substitution to find the solutions.
- The matrix changes wherever we perform an elementary row operation, so that we cannot use equal signs between the steps. Use a $\sim$ instead. The matrices are said to be (row-)equivalent.
- We can proceed in a similar manner to obtain '0' above the pivots too, the reduced row-echelon matrix. This is called *Gauss-Jordan elimination*. The solution can be read directly from the final matrix, without the need for back substitution.

2.2 Using the calculator

The calculator program GAUSS automates the Gauss-elimination procedure, while the GAUSSJDN program automates the Gauss-Jordan-elimination procedure.

Using the programs

Store the matrix in Mat A using the MENU MAT icon. Select the MENU PRGM icon Run the program to see either the row-echelon form of Mat A (GAUSS) or the reduced-row-echelon form of Mat A (GAUSSJDN). These are stored in Mat Ans and Mat R if you want to use them in further calculations.

Examples: Gauss elimination

$$\begin{bmatrix} 10 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & .4 & .1 & .1 \\ 0 & 1 & -1 & -8.5 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2/5 & 1/0 & 1/10 \\ 0 & 1 & -1 & -17/2 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & .\dot{3} & .1\dot{6} & .\dot{6} \\ 0 & 1 & .25 & .25 \\ 0 & 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & 1/6 & 2/3 \\ 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & .25 & .25 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1/4 & 1/4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 5 & 1 & 3 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 Solving Systems of Linear Equations

Work through the matrix operations in the first section of these notes, if you haven't already done so.

1. Enter the matrix below carefully into Mat A using MENU MAT.

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

2. Mat A is the augmented matrix $[\boldsymbol{P}|\boldsymbol{Q}]$ for the following system of equations: in matrix form $\boldsymbol{PX}=\boldsymbol{Q}$.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

Solve this system

(a) using the calculator to find the row-echelon form (Gauss elimination), and then back substitution (you have to do this) to find the answer.

(b) using the calculator to find the reduced row-echelon form (Gauss-Jordan elimination), from which you can read off the answer.

(c) by using $\boldsymbol{X}=\boldsymbol{P}^{-1}\boldsymbol{Q}$. *Do you understand why this works?*

A little manipulation on the calculator allows us to evaluate $\boldsymbol{P}^{-1}\boldsymbol{Q}$. Press [MENU] [3], make sure the cursor is on Mat A and press [EXE]. Change Mat A to the matrix $\boldsymbol{P}$ by pressing [F3] (COL), moving the cursor to Column 5 and pressing [F1] (DEL). Press [EXIT] twice and store the 4×1 matrix $\boldsymbol{Q}$ (the right-hand sides of the equations) in Mat B. Press [MENU] [1] to return to the RUN screen and evaluate Mat A^{-1} Mat B.

3. Solve the systems of equations over the page using each of the three methods. Check your answers by substituting them back into the equations. Can you use Method 2(c) to solve these? What happens in Examples (b) and (c) below? Can you explain? What is $\det(\boldsymbol{P})$? What does this tell you about the inverse matrix?

(a)

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ x - 2y + 3z &= 9 \\ -x + 3y &= -4 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

(c)

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

4 Eigenvalues and Eigenvectors

Given an $n \times n$ matrix $\boldsymbol{A}$, the eigenvalues of $\boldsymbol{A}$ are the constants λ and the eigenvectors of $\boldsymbol{A}$ are the corresponding n-dimensional vectors $\boldsymbol{X}$ ($n \times 1$ matrices) satisfying the equation

$$\boldsymbol{A}\boldsymbol{X} = \lambda \boldsymbol{X}.$$

The eigenvalues λ are the zeros or roots of the nth-degree characteristic polynomial

$$p(x) = \det\bigl(\boldsymbol{A} - x\boldsymbol{I}\bigr),$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix.

A nice way to find the eigenvalues of a 3×3 or 2×2 matrix[2] on a CFX-9850 is graphically. With your matrix in Mat A, run the EIGENVAL program to graph the characteristic polynomial. To find the roots of the polynomial, press [MENU] [5] (GRAPH), [F5] to redraw the graph and [F5] again to select G-Solv. Press [F1] (ROOT) to find the first root of the polynomial and the right arrow successively to find the other roots: these are the eigenvalues λ_i, $i = 1, 2, \ldots$.

To find the eigenvectors $\boldsymbol{X}$, we have to solve the homogeneous matrix equation

$$\bigl(\boldsymbol{A} - \lambda_i \boldsymbol{I}\bigr)\boldsymbol{X} = \boldsymbol{0}$$

for each eigenvalue λ_i found above. We do this using Gauss or Gauss-Jordan elimination, most simply done using the GAUSS or GAUSSJDN program on the matrix $\boldsymbol{A} - \lambda_i \boldsymbol{I}$ for each eigenvalue λ_i, with $\boldsymbol{A} - \lambda_i \boldsymbol{I}$ stored in Mat A.

Hint: Store the original matrix $\boldsymbol{A}$ in Mat B. Then the combinations $\boldsymbol{A} - \lambda_i \boldsymbol{I}$, in calculator terms Mat B $-\ \lambda_i$identity 3, can be stored successively in Mat A for use in the program.

Example

Find the eigenvalues and corresponding eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

This is a simple problem that can be done easily by hand. We use it to illustrate the calculator method, which can be used for more complicated problems.

[2]Make your 2×2 matrix into a 3×3 by putting all zeroes in Row 3 and Column 3. One of the eigenvalues will always be 0 — the other two eigenvalues are the eigenvalues of the 2×2 matrix.

Eigenvalues

Put $\boldsymbol{A}$ in your calculator in the form

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

and run EIGENVAL.

The program plots the characteristic polynomial, here it is clear that the roots lie in the range $0 \leq x < 3$. The root at 0 is a spurious one, introduced by extending the 2×2 matrix to a 3×3. Using *root* in the G-Solv menu gives the eigenvalues as $\lambda_1 = 1$ and $\lambda_2 = 2$.

In this simple case, it is easy to show by hand that the characteristic polynomial is $p(\lambda) = \lambda^2 - 3\lambda + 2$, with zeroes 1 and 2.

Eigenvectors

$\boldsymbol{\lambda_1 = 1}$: We have to find $\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, such that $\boldsymbol{AX}{=}\boldsymbol{X}$, i.e. solve

$$\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

or equivalently

$$\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

To use the GAUSS program, we need to put $\boldsymbol{A}{-}\boldsymbol{I}$ in Mat A. Since we want to use $\boldsymbol{A}$, now in Mat A, again later, store Mat A in Mat B. Then store Mat B − identity 3 in Mat A and run GAUSS to give the equivalent row-echelon form of $\boldsymbol{A}{-}\boldsymbol{I}$ as

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeroes. There must be at least one row of zeroes (the bottom row) in all eigenvector problems.

The corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The bottom row tells us that x_2 is arbitrary, so we set $x_2 = t$, where t is any number. The top row tells us that $x_1 + x_2 = 0$, so that $x_1 = -x_2 = -t$. The eigenvector $\underset{\sim}{\boldsymbol{v}}_1$ corresponding to $\lambda_1 = 1$ is therefore

$$\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero[3]) multiples of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We usually say that this vector is an eigenvector, with the understanding that all non-zero multiples of it are also eigenvectors.

Check:

$$\boldsymbol{AX} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \boldsymbol{X}.$$

$\boldsymbol{\lambda_2 = 2}$: We have to solve $\boldsymbol{AX} = 2\boldsymbol{X}$ or $\big(\boldsymbol{A} - 2\boldsymbol{I}\big)\boldsymbol{X} = \boldsymbol{0}$. Therefore, we store Mat B −2 identity 3 in Mat A and run GAUSS to give the equivalent row-echelon form of $\boldsymbol{A} - 2\boldsymbol{I}$ as

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $x_2 = t$, where t is any number, and $x_1 + 2x_2 = 0$, so that $x_1 = -2x_2 = -2t$. The eigenvector $\underset{\sim}{\boldsymbol{v}}_2$ corresponding to $\lambda_2 = 2$ is therefore

$$\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The eigenvector is an arbitrary (non-zero) multiple of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Check:

$$\boldsymbol{AX} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2\boldsymbol{X}.$$

[3]If $t = 0$, we obtain a zero vector, which, by definition, cannot be an eigenvector.

Exercise: Show that the eigenvalues of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

are -1, 1 and 2, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

5 Answers

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2. $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$.

3. **(a)** $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

(b) $x_1 = 1 + 2t$, $x_2 = 2 + 3t$, $x_3 = t$; t any non-zero number.

(c) No solution.

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The eigenvalues are -1, 1 and 2, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$