Sequences and Series on a CFX-9850GB

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Contents

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- *Matrices on an CFX-9850GB* — suitable for Years 11 and 12.
- *Calculus on an CFX-9850GB* — suitable for Years 11 and 12.
- *Complex Numbers on a CFX-9850* — suitable for Year 12.
- *Introduction to Complex Numbers* — complex numbers from the beginning, covering the basic operations, but set in the context of complex numbers as a mathematical structure.

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1 Introduction

These notes provide a comprehensive review of generating, displaying and graphing sequences and series on a TI-83/83+ graphics calculator. An arithmetic progression, a geometric progression and the Fibonacci numbers are used as examples. A number of questions (with solutions) illustrate the use of the calculator. Finally there are three topics that could be used as a basis for group investigation or a small project.

The CFX-9850G PLUS can generate sequences, sum series, and display sequence terms in a table or graph. However, we should first ask whether it makes sense to use a graphics calculator at all for sequences and series.

Certainly the first few lessons on sequences should be pencil and paper, until some of the concepts and calculations are understood, although a class activity such as that on page 7 can add variety to the early learning stages. However, having to work out terms of a sequence or series by hand eventually becomes tedious, especially those terms that are not very simple and require a calculator anyway. This becomes an impediment to further learning and exploration.

The calculator automates the process of calculating terms in a sequence or series *once it is given an appropriate definition*. It is in finding an appropriate definition that most of the thought goes — the calculator can't do this. With automatic calculation comes the ability to explore particular sequences and series, to conjecture and test, and to look at ideas such as the convergence of an infinite sequence or series.[1]

Some of the questions and investigations in Sections 4 and 5 demonstrate this extra capability when using a graphics calculator.

[1]These are not esoteric beasts — the humble AP and GP continue on indefinitely.

2 Sequences

A sequence is an ordered set of numbers, usually with the numbers or terms in the sequence determined by some sort of formula.[2]

For example

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \ldots \qquad 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \ldots \qquad 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \ldots$$

are sequences for which we can find a formula to determine each term.

In the usual notation, a general sequence is written as

$$a_1, a_2, \ldots, a_n, \ldots,$$

where each term , a_1, a_2, $a_3, \ldots$ is a number. The subscript gives the position of the term in the sequence.

There are two ways to give a formula for each term.

- **Recursively:** write the nth term in terms of the previous term or terms. Here we also need to know a value for the first term (or first few terms) in the sequence.

  Examples

  1. $a_2 = a_1 + 2$, $a_3 = a_2 + 2, \ldots$ or in general, $a_{n+1} = a_n + 2$. With $a_1 = 1$, this recursive formula gives the first sequence above.
  2. $a_{n+1} = a_n/2$, $a_1 = 1$ gives the second sequence above.
  3. $a_{n+2} = a_{n+1} + a_n$, $a_1 = 1$, $a_2 = 1$ gives the third sequence above, the famous Fibonacci numbers.

- **Explicitly:** specify the nth term as a function of n, where n takes integer values.

  Examples

  1. $a_n = 2n - 1$, $n = 1, 2, 3, \ldots$, which again gives the first sequence above.
  2. $a_n = 0.5^{n-1}$, $n = 1, 2, 3, \ldots$, which again gives the second sequence above.

  The Fibonacci sequence can also be defined explicitly — see Section 5.

There are basically two ways on the CFX-9850 to generate and display terms of a sequence — *using the RECUR mode* and *using LIST commands in RUN mode*. Which method you use will depend on how you teach the topic. Here we'll look at both, using the above examples to illustrate the methods. There are questions to practise on in Section 4.

[2] although we can have sequences of random numbers.

2.1 Using RECUR mode

Sequences can be defined either recursively or explicitly, displayed as a table and graphed in RECUR mode. Press [MENU] [8] or move the cursor to the RECUR icon and press [EXE].

If your screen does not look like the one above, press [F3] (TYPE) and [F2].

2.1.1 Arithmetic progression

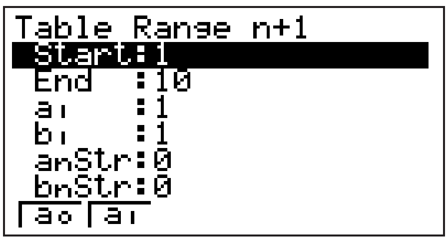
In an arithmetic progression, there is a constant difference between successive terms. The recursive definition is $a_{n+1} = a_n + d$, where d is a constant called the common difference.

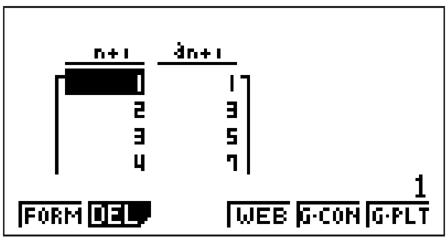
Example: $a_{n+1} = a_n + 2$, with $a_1 = 1$.

Set this sequence up on your calculator as shown below. Press [F4] to show the keys for n and a_n. Press [EXE] to store the definition. The first term a_1 is input when we set up the table below.

Displaying the sequence

Press [F5] (RANG) and set your screen up like the one below, pressing [EXE] after each entry. [F1] and [F2] allow you to choose whether to start the sequence at a_0 or a_1. Press [EXIT] when you have finished and [F6] (TABL) to display the table. [F1] (FORM) takes you back to the formula for a_{n+1} if you need to change it.





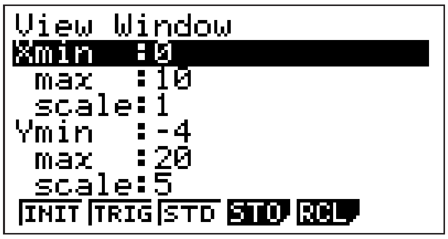
Graphing the sequence

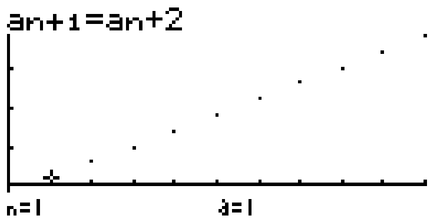
There are several ways to graph sequences, selected by F4, F5 and F6.

- F4 (WEB) plots a cobweb plot, a_{n+1} versus a_n. We'll look at this in Section 5.2.
- F5 (G·CON) plots a so-called time plot, a_n versus n, connecting the points.
- F6 (G·PLT) also plots a_n versus n, but doesn't connect the points. This is what we will use here after setting the view window.

To set the view window, press SHIFT F3. In our time plot, n is plotted along the X axis and a_n along the Y axis. Set up your view window as shown below and press EXIT, F6 (TABL) and F6 again (G·PLT). Press F1 (TRACE) and use the arrow keys to move along the points of the sequence.

The value of -4 for Ymin is to allow space for the coordinates at the bottom of the screen when using TRACE.





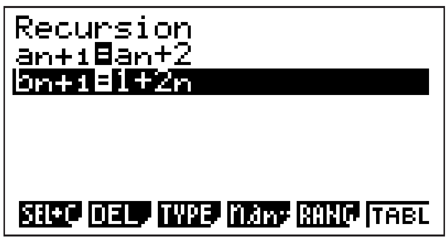
Defining the AP explicitly

The nth term for a general AP is $a_n = a_1 + (n-1)d$, or using the calculator notation, $a_{n+1} = a_1 + nd$, where a_1 is the first term and d is the common difference.

For our example, we have $a_1 = 1$ and $d = 2$, so that

$$a_{n+1} = 1 + 2n.$$

We can compare the two definitions of the sequence by putting the nth term in sequence b, as shown below.[3]



Now press F6 and compare the sequences a and b. You should see that the two sequences are identical. If the first terms are different, go back to RANG and make sure $b_1 = 1$.

[3] We could have chosen F1 from the TYPE menu to enter the formula for a_n rather than a_{n+1}, but we couldn't then compare the two sequence definitions.

2.1.2 Geometric progression

In a geometric progression, each term is a constant multiple of the previous term. In calculator notation, the recursive definition is

$$a_{n+1} = ra_n,$$

where r is a constant called the common ratio or common multiplier.

Exercise: Display a table and graph the first ten terms of the geometric sequence $a_{n+1} = 0.5a_n$, with $a_1 = 1$.

The nth term of a geometric progression is given *explicitly* by

$$a_{n+1} = cr^n,$$

where c is a constant and r is the common multiplier.

For the sequence in the exercise above, the nth term is given by

$$a_{n+1} = 0.5^n.$$

Exercise: Put this explicit definition in sequence b and compare with the recursive definition. Make sure the value for b_1 in RANG is correct.

2.1.3 Fibonacci sequence

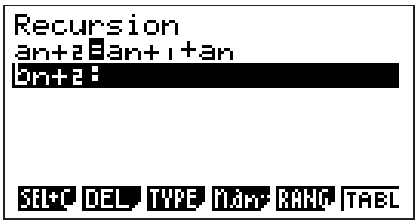
The Fibonacci sequence is defined, in calculator terms, by

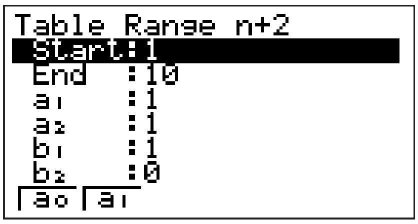
$$a_{n+2} = a_{n+1} + a_n,$$

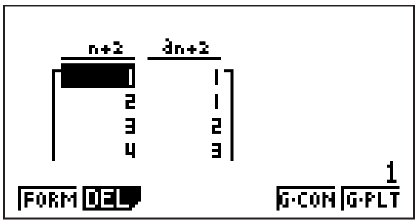
with $a_1 = 1$ and $a_2 = 1$.

Press [MENU] [8], use [F3] (TYPE) and [F3] again to select the appropriate form. Enter the Fibonacci formula.

Set the two starting values $a_1 = 1$ and $a_2 = 1$ in RANG. Generate a table of values.







2.2 Using LIST commands in RUN mode

Press MENU 1 to return to RUN mode. Press OPTN to bring up menus at the bottom of the screen, the first of which is LIST. Press F1 to select this.

The *Seq* command (F5) in the LIST menu generates a sequence (list) specified by an explicit general term. The syntax is

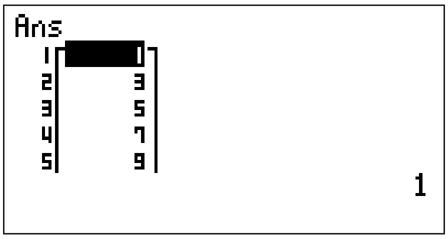
Seq(general term, variable, start, end, step),

where variable can be any letter.[4] The total number of terms calculated must be less than 256.

2.2.1 Arithmetic progression

In the following sections, there is an implicit EXE after each command.

Seq(1 + 2(N−1), N, 1, 10, 1)



Scroll down/up the sequence using the arrow keys. Press EXIT to return to the RUN screen.

2.2.2 Storing to a list

For analysis and graphing, it is often convenient to store a sequence in a named list, which is then stored in memory. The CFX-9850 has six built-in lists: List 1 – List 6.

To store a sequence to List 1,

Seq(2N−1, N, 1, 10, 1) → List 1,

where *List* is F1 in the LIST menu.

Now press MENU 4 (LIST) to enter the list editor. You can scroll down, edit and carry out other operations here.

[4]The variable in a *Seq* command doesn't have to take integer values, for example *Seq(sin(X), X, 0, 1, 0.05)* generates a sequence of values of the sine function.

2.2.3 Geometric progression

Seq(0.5∧(N−1), N, 1, 10, 1)

Exercise: Use the left or right arrow to recall the previous command and edit it to produce terms 11 to 20 in the sequence.

Class Activity

Give students one calculator between two.

Set up the OHP calculator to evaluate sequences of the form $a_n = An + B$ (F1 in TYPE in RECUR mode).

On the OHP calculator, with the OHP turned off, set $a_n = 2n + 1$. Set RANG to generate 10 terms in the sequence, press EXIT and then TABL. Turn the OHP on and ask the students: *What's the rule?*

When they have worked it out as a class, show them how the rule is entered and RANG is set. Have them enter the rule and generate the table. Write on the board that the rules to follow are all of the form __n + __.

Now, again with the OHP off, enter a different rule (remember to press EXE) and show them the table. Ask them to make the table on their calculators the same. Remember FORM and TABL are the two keys to move between table and formula.

Give them various rules to find, moving eventually to negative numbers for the coefficients. Ask them to summarise their findings regarding the two numbers in the rules.

3 Series

A series is the sum of the terms in a sequence, that is a sequence with + signs between successive terms. However, we also say ‘the sum of a series’ to distinguish the sequence with + signs from the actual value when we carry out the additions. To calculate the sum of a series, we need to find the terms in the corresponding sequence, then add them up.

If a series has a finite number of terms, we just add them up to give the sum. All of what we do below can be applied to finite series. However, the more interesting series are infinite — we can’t calculate the sum for these by carrying out the additions because there is an infinite number of them.[5] However, we can work out the sum of a finite number of terms, called a partial sum — the nth partial sum is the sum of the first n terms of the series. The behaviour of the partial sums as n gets bigger tells us something about the convergence of the series — whether the sum may be a finite number or infinite. The sum of an infinite series is defined as the limit of its partial sums as $n \to \infty$.

Both RECUR mode and the LIST commands can be used to sum a series, but only RECUR mode works for terms defined recursively.

3.1 Using RECUR mode

The nth partial sum of a series S_n is the sum of the previous $n-1$ terms, S_{n-1}, plus the nth term; partial sums can therefore be defined recursively. In symbols,

$$S_n = S_{n-1} + a_n \qquad S_1 = a_1.$$

We’d like to generate the sequence of partial sums

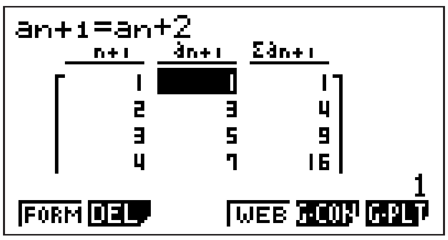
$$S_2, S_3, S_4, S_5 \ldots.$$

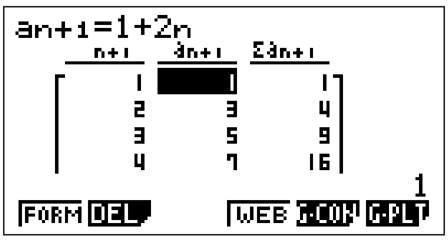
Enter RECUR mode (MENU 8).

The CFX-9850 will calculate the partial sums automatically: press SHIFT MENU and turn on Σ Display. Press EXIT to return to the formula-entry screen.

3.1.1 Arithmetic progression

Displaying the table for the AP with general term $a_{n+1} = a_n + 2$ or $a_{n+1} = 1 + 2n$ also displays the partial sums.





[5] There are alternative methods for summing some infinite series, one of the triumphs of Calculus.

3.1.2 Geometric progression

Exercise: Set up a table of partial sums for our GP with $a_{n+1} = 0.5a_n$, $a_1 = 1$ or $a_{n+1} = 0.5^n$. To what value does the (infinite) series appear to converge. Confirm your answer algebraically.

3.2 Using LIST commands in RUN mode

The *Sum* command in the LIST menu (press F6 twice) sums a sequence (list). The syntax is

Sum(*list*),

where *list* can be a *Seq* command.

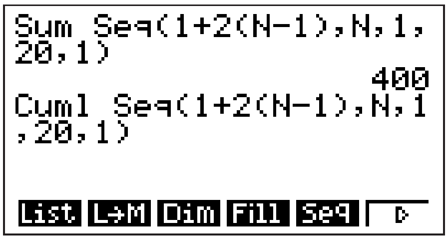
The *Cuml* command in the LIST menu generates a sequence of partial sums. The syntax is

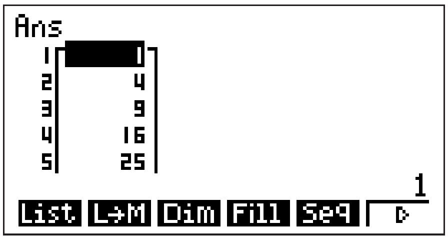
Cuml(*list*).

3.2.1 Arithmetic progression

Sum(Seq(1 + 2(N−1), N, 1, 20, 1)) finds the sum of the first 20 terms in our AP.

Cuml(Seq(1 + 2(N−1), N, 1, 20, 1)) generates the first 20 partial sums of our AP: the ith entry is the sum of the first i terms.





The sum of an infinite series is defined as the limit as $n \to \infty$ of the nth partial sum. Scrolling down a list of cumulative sums like the one above can give an idea of what that limit might be. Here, of course, there is no limit — the nth partial sum $\to \infty$ as $n \to \infty$.

3.2.2 Geometric progression

Sum(Seq(0.5∧(N−1), N, 1, 20, 1))

Cuml(Seq(0.5∧(N−1), N, 1, 20, 1))

Exercise: What's the sum of the GP $\quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$?

You might want to recall the command you enter (left or right arrow) and increase the end value of N to be (almost) sure of the answer. We can't *prove* this is the answer numerically, but we can be reasonably sure it is the answer.

4 Questions on sequences and series

1. Generate a sequence of the cubes of first ten positive integers using a LIST command. Store the sequence in a list. Use the *Sum* command to evaluate $1^3 + 2^3 + \cdots + 10^3$.

2. Find the sum of the first ten terms of the series $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots$.

3. (a) Use a single command to find the sum of the first 100 positive integers. Store the answer in memory S for use in (c).

(b) Edit your command in (a) (left or right arrow) to find the sum of the *cubes* of the first 100 positive integers.

(c) Which is bigger: the sum of the cubes of the first 100 positive integers or the cube of the sum of the first 100 positive integers?

4. The half-life of a certain radioactive substance is one week. This means that, of the amount present at a particular time, only half will be left a week later. Suppose 1000 grams of the substance exists today, the beginning of Week 1.

(a) Write down the amount left at the beginning of Week 2, Week 3, ..., Week 10.

(b) Determine an infinite geometric sequence (recursive or explicit) that is a model of the amount of the substance at the beginning of Week n, where $n = 1, 2, 3, \ldots$. What is the common ratio of this sequence?

(c) When will there be only 0.005 grams remaining?

(d) How much of the substance was there a week ago (beginning of Week 0)?

(e) When will the substance be reduced to nothing *according to this model*?

5. The height of a particular fast-growing plant increases at the rate of 2.5% per month. Assume the plant is 30 cm high today and that it dies after 12 months.

(a) Determine a finite geometric sequence that is a model of the height of the plant after n months. Write out all the terms of the sequence. What is the common ratio?

(b) How long would the plant have to live to double in height??

6. Sue had $1250 in a savings account three years ago. What will be the value of her account two years from now, assuming no deposits or withdrawals are made and the account earns 6.5% interest compounded annually?

7. Frank has $12,876 in a savings account today. He made no deposits or withdrawals during the last six years. What was the value of his account six years ago? Assume that the account earned 5.75% interest compounded monthly.

8. Generating sequences recursively is equivalent to another mathematical process called iteration, in which we do the same operation over and over. Try some of the following sequences/iterations.

Generate some terms in each sequence. What happens to a_n in each sequence as n becomes large?

(a) $a_{n+1} = a_n^2$. Try a_1 greater than 1; $a_1 = 1$ and -1; a_1 between -1 and 1; a_1 less than -1.

(b) $a_{n+1} = a_n^2 - 1$ with values of a_1 between -2 and 2. Note that $\frac{1}{2}(1+\sqrt{5}) = 1.618\ldots$ and $\frac{1}{2}(1-\sqrt{5}) = 0.618\ldots$. What's special about these values for a_1?

(c) $a_{n+1} = \sqrt{a_n}$.

(d) $a_{n+1} = \cos(a_n)$.

(e) $a_{n+1} = \tan(a_n)$.

See *Mathematics by Computer: Iteration* by Lynne Kelly, Wizard Books, 1996 for more on iteration.

9. What value does the sequence $\left(1 + \dfrac{1}{n}\right)^n$ approach as n gets larger and larger?

Hint: Use a sequence command with a step of at least 1000 and at least 50 terms.

10. What is $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$? factorial ! is in the OPTION PRB menu.

The *Cuml Seq* combination works well here, as we don't need too many terms to see the convergence of the partial sums. Note that the series starts at $n = 0$.

What is $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$? $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, where x is any number?

The following questions are from Chapter 9 of *Intermediate Algebra: Functions and Graphs* by K. Yoshiwara and B. Yoshiwara, Thomson Brooks/Cole, 2004.

11. Generate a table of values for each of the following recursive sequences. What happens to the terms as n gets larger? Do you recognise the number?

Hint: The answers for the first two sequences are square roots of particular numbers. Can you make and test a conjecture here? Write down a sequence whose limit is 2; $\sqrt{5}$.

The answers for the last two sequences should be obvious. Can you make and test a conjecture here? Write down a sequence whose limit is 4; 5.

(a) $a_1 = 1 \quad a_{n+1} = \dfrac{1}{1+a_n} + 1$ (b) $b_1 = 1 \quad b_{n+1} = \dfrac{2}{1+b_n} + 1$

(c) $s_1 = 1 \quad s_{n+1} = \dfrac{1}{2}\left(s_n + \dfrac{4}{s_n}\right)$ (d) $t_1 = 1 \quad t_{n+1} = \dfrac{1}{2}\left(t_n + \dfrac{9}{t_n}\right)$

12. A rubber ball is dropped from a height of 8 metres and returns to three-quarters of its previous height on each bounce.

(a) How high does the ball bounce after hitting the floor for the third time? for the tenth time?

(b) How far has the ball travelled vertically when it hits the floor for the fourth time? for the twentieth time?

13. According to legend, a man who had pleased the Persian king asked for the following reward. The man was to receive a single grain of wheat for the first square of a chess-board, two grains for the second square, four grains for the third square, and so on, doubling the amount for each square up to the 64th square. How many grains would he receive in all. (Fortunately the king had a good sense of humour.)

14. Find the sum of all integral multiples of 6 between

(a) 10 and 100.

(b) between 1 and 10,000.

5 Further investigations

5.1 The Fibonacci sequence and the Golden Ratio

Here is one illustration of the many interesting properties of the Fibonacci sequence. For more, see the web site *www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html.*

1. Generate a table of values of the sequence (be very careful with brackets)

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Do you recognise this sequence?

2. One of the many interesting things about the Fibonacci sequence becomes apparent when we look at ratios of successive terms. Using RECUR mode, set

$$a_{n+2} = a_{n+1} + a_n \qquad a_1 = a_2 = 1$$

to generate the Fibonacci sequence. Set

$$b_{n+2} = a_{n+1}/a_n$$

to calculate the ratios of successive terms of the Fibonacci sequence.

Generate a table of values of 30 terms.

Scroll down and look at the b_{n+2} column containing the ratios of successive terms of a_{n+2}. Does this sequence appear to be approaching a particular value? What value? Find this value accurate to six decimal places.

Now find a value for the Golden Ratio or Golden Section, $\dfrac{1+\sqrt{5}}{2}$.

What conclusion do you reach?

5.2 The logistic sequence

The logistic sequence or logistic map has become famous because it is one of the simplest sequences that exhibits chaotic behaviour. It also turns up in a number of areas such as population modelling.[6]

The logistic sequence is defined by

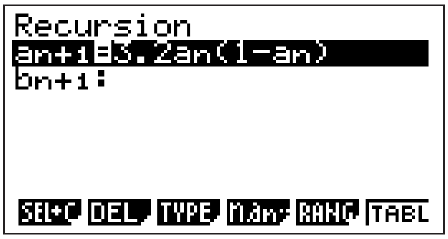
$$a_{n+1} = Aa_n(1 - a_n),$$

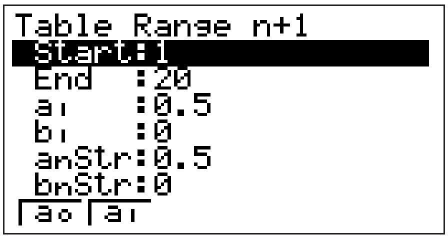
where A is a constant.
The following figures show the set up for *time plots*, a_n vs n, on the left and *cobweb plots*, a_{n+1} vs a_n, on the right. In both sets of plots $a_1 = 0.5$; for web plots, we must set anStr in RANG (Table Range) to the same value. Put the value for A in the formula for a_{n+1}. The screen below has $A = 3.2$.

Set the view window (SHIFT F3) appropriate to the type of graph, press EXIT then TABL and either F5 or F4 to generate the graph. Select F1 (a_n) if there is a choice (when Σ Display is on).

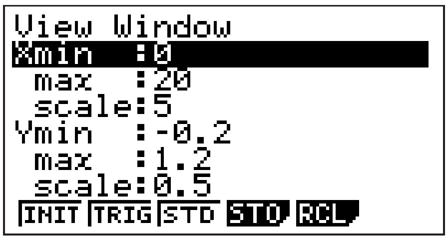
For a time plot, press F1 (TRACE) so you can trace the graph.

For a cobweb plot, keep pressing EXE to generate the plot. If nothing happens, you may not have set anStr in RANG to the correct value. In a cobweb plot, the curves $y = Ax(1 - x)$ and $y = x$ are also plotted. The cobweb lines move between these two curves.

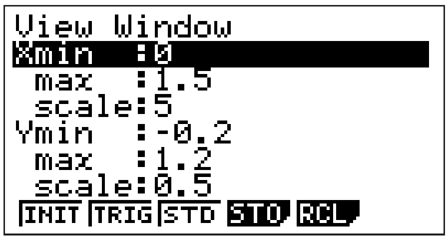


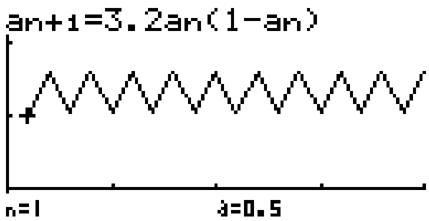


TIME **WEB**

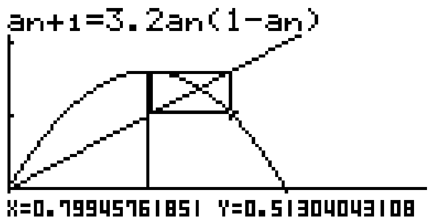


WINDOW

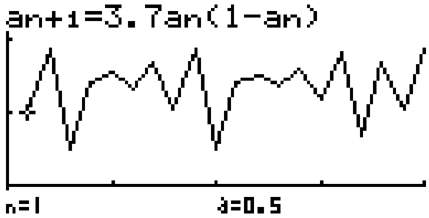




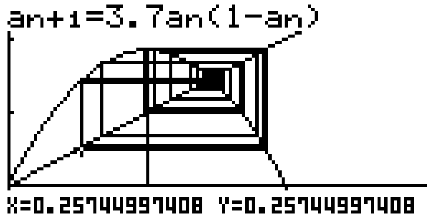
A = 3.2



[6]For examples, see Sections 2.6, 2.7 of *Population Modelling* at *www.ma.adfa.edu.au* under *High School and College Activities*.



A = 3.7



Confirm graphically (by choosing appropriate values for A) that the sequence converges to 0 for $0 < A < 1$; to $1 - 1/A$ for $1 < A < 3$; and oscillates between 2 values, then 4 values, 8 values and so on as A is increased above 3. It becomes chaotic for values of A greater than about 3.568. Try both time and web plots.

For values of A greater than 4 or values of a_1 less than 0 or greater than 1, the sequence diverges rapidly to $\pm\infty$.

The book *Mathematics by Computer: Iteration* by Lynne Kelly, Wizard Books, 1996 has some good worksheets and discussion on iteration in general, including a section on *The Logistic Equation*.

6 Solutions to questions

1. The command is Seq (X∧3, X, 1, 10) → List 1. You can use any letter instead of X: in the text we used N to conform to the standard notation for sequences. X is easiest to use because it requires only one key press.

To calculate the sum, use Sum List 1 or Sum Seq (X∧3, X, 1, 10, 1) to produce the answer 3025.

2. The command is Sum Seq (1/X∧3, X, 1, 10, 1), giving 1.197531986.

3. (a) The sum of the first 100 positive integers is Sum Seq (X, X, 1, 100, 1) = 5050.

(b) The sum of the *cubes* of the first 100 positive integers is Sum Seq (X∧3, X, 1, 100, 1) = 25,502,500.

(c) The cube of the sum of the first 100 positive integers is $5050^3 \approx 1.29 \times 10^{11}$, larger than the sum of the cubes of the first 100 positive integers, which is only about 2.55×10^7.

4. (a) The amount of radioactive substance in grams at the beginning of successive weeks, starting at Week 1, is (to two decimal places)

$$1000,\ 500,\ 250,\ 125,\ 62.5,\ 31.25,\ 15.63,\ 7.81,\ 3.91,\ 1.95.$$

(b) Explicitly, the amount at the beginning of Week n is $a_n = 1000\big(0.5\big)^{n-1}$. Recursively, $a_{n+1} = 0.5a_n$, with $a_1 = 1000$. The common ratio here is 0.5.

(c) Using a calculator table with $a_n = 1000\big(0.5\big)^{n-1}$, there is 0.005 g remaining sometime in Week 18, i.e. between $n = 18$ and $n = 19$.

(d) A week ago (beginning of Week 0), there was twice as much as there is now (beginning of Week 1), that is 2000 g.

(e) According to this model, there will always be some of the substance left, although the amount becomes small very rapidly. You can't reduce any number to 0 by dividing it by 2 or raising it to a power.

5. (a) The height n months from now is $a_n = 30\left(1 + \frac{2.5}{100}\right)^n = 30\big(1.025\big)^n$, $n = 1, 2, \ldots, 12$. The table below shows months (top row) and corresponding heights, rounded to one decimal place.

0	1	2	3	4	5	6	7	8	9	10	11	12
30	30.8	31.5	32.3	33.1	33.9	34.8	35.7	36.6	37.5	38.4	39.4	40.4

The common ratio is 1.025.

(b) Using the calculator table, the plant would double in height in the 29th month.

6. Let S_n be the amount of money in Sue's account at the start of year n, with $n = 1$ corresponding to 3 years ago. Then,

$$S_n = 1250\left(1 + \frac{6.5}{100}\right)^{n-1} = 1250\big(1.065\big)^{n-1}.$$

In two year's time, $n = 1 + 5 = 6$, so the amount of money in her account will be

$$S_5 = 1250\big(1.065\big)^5 = \$1712.61 \quad \text{to the nearest cent.}$$

You could also use the calculator table to reach this answer.

7. The amount in Frank's account is given by

$$S_n = 12876\left(1 + \frac{5.75}{12 \times 100}\right)^{12(n-1)} = 12876\left(1 + \frac{0.0575}{12}\right)^{12(n-1)},$$

where $n = 1$ corresponds to now.

Six years ago, $n = -5$, and the amount in Frank's account was

$$S_{-5} = 12876\left(1 + \frac{0.0575}{12}\right)^{-72} = \$9126.56 \quad \text{to the nearest cent.}$$

You could also use the calculator table to reach this answer. Start the table at -5. Do you get the correct value at $n = 1$?

8. (a) Set $a_{n+1} = a_n^2$, set a_1 to an appropriate starting value and use the calculator table to see what happens as n becomes large.

Initial value	Limit of sequence
$a_1 > 1$	∞
$a_1 = 1$	1
$-1 < a_1 < 1$	0
$a_1 = -1$	1
$a_1 < -1$	∞

(b) Set $a_{n+1} = a_n^2 - 1$.

Initial value	Limit of sequence
$a_1 < -1.618\ldots$	∞
$a_1 = -1.618\ldots$	$-1.618\ldots$ (constant)
$-1.618\cdots < a_1 < -0.618\ldots$	oscillates between 0 and -1
$a_1 = -0.618\ldots$	$-0.618\ldots$ (constant)
$-0.618\cdots < a_1 < 0.618\ldots$	oscillates between 0 and -1
$a_1 = 0.618\ldots$	$-0.618\ldots$ (constant)
$0.618\cdots < a_1 < 1.618\ldots$	oscillates between 0 and -1
$a_1 = 1.618\ldots$	$1.618\ldots$ (constant)
$a_1 > 1.618\ldots$	∞

(c) $a_{n+1} = \sqrt{a_n}$ gives 0 if $a_1 = 0$, and tends to 1 otherwise.

(d) $a_{n+1} = \cos\left(a_n\right)$ tends to 0.7391... in radian MODE and 0.99985... in degree MODE.

(e) $a_{n+1} = \tan\left(a_n\right)$ looks several times as though it is going to settle down to a limit, but never does.

9. The sequence $\left(1 + \dfrac{1}{n}\right)^n$ approaches the value $e = 2.71828\ldots$ as n goes to ∞.

10. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} = e \qquad \sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2 \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$

11. (a) approaches $\sqrt{2}$; (b) approaches $\sqrt{3}$; (c) approaches 2; (d) approaches 3.

12. (a) The following table gives the rebound height after bounce n.

n	1	2	3	4	...
Height (m)	6	$\frac{9}{2}$	$\frac{27}{8}$	$\frac{81}{32}$	...

The rebound height after the third bounce is therefore 27/8 m.

The rebound height after bounce n is $6\left(0.75\right)^{n-1}$ m. Therefore, after the tenth bounce, the rebound height is $6\left(0.75\right)^9 \approx 0.45$ m. Alternatively, use a table.

(b) The total distance travelled is 8 + 2 × each rebound height, i.e.

$$8 + 2\left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32} + \ldots\right).$$

When the ball hits the floor for the fourth time, it has travelled

$$8 + 2\left(6 + \frac{9}{2} + \frac{27}{8} + \frac{81}{32}\right) = 35.75\,\text{m}.$$

The distance travelled when it hits the floor for the bth time ($b \geq 2$) is therefore

$$8 + 2\sum_{n=1}^{b-1} 6\big(0.75\big)^{n-1} = 8 + 12\sum_{n=1}^{b-1} 0.75^{n-1}.$$

Therefore, when the ball hits the floor for the twentieth time, it has travelled

$$8 + 12\sum_{n=1}^{19} 0.75^{n-1} \approx 55.8\,\mathrm{m}.$$

Here, use the command 8 + 12 Sum Seq(0.75∧(B−1), B, 1, 19, 1)) to do the calculation.

13. Number of grains $= 1 + 2 + 4 + 8 + 16 + \ldots = \displaystyle\sum_{n=1}^{64} 2^{n-1} \approx 1.84 \times 10^{19}$.

Use Sum Seq(2∧(N−1), N, 1, 64, 1) to calculate this.

To put this number into context, 10^{18} grains would be about the same volume as the Great Wall of China and the five Great Pyramids combined.[7]

14. We need to evaluate $\displaystyle\sum_{n=a}^{b} 6n$, where

(a) $6a$ is the smallest multiple of 6 greater than 10 and $6b$ is the largest multiple of 6 less than 100. Clearly, $a = 2$ and $b = 16$, so that the required sum is

$$\sum_{n=2}^{16} 6n = 810,$$

where we use the command Sum Seq(6N, N, 2, 16, 1).

(b) $6a$ is the smallest multiple of 6 greater than 1 and $6b$ is the largest multiple of 6 less than 10,000. Clearly, $a = 1$ and b is the integer part of 10,000/6, that is 1666. The required sum is

$$\sum_{n=1}^{1666} 6n = 8,331,666.$$

We can't use the single command Sum Seq(6N, N, 1, 1666, 1) here because there are more than 255 terms in the series. We must break the sum up into parts: Sum Seq(6N, N, 1, 255, 1); Sum Seq(6N, N, 256, 510, 1); Sum Seq(6N, N, 511, 765, 1); and so on.

[7]From *Large Numbers* by Victor Scharaschkin, Australian Senior Mathematics Journal 4 (2), 111–125 (1990).