Calculus on an EL-9650/9900

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Contents

ADFA **CMA**

Contact for more help

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At *www.unsw.adfa.edu.au/pems/news/high_school/hsc_activities.html*

- A variety of graphics-calculator activities for Years 9 and 10 — written as part of the CQTP Program for Sharp, Casio and TI calculators.
- *Using the EL-9650/9900* — an introduction to the basic operations, suitable for Years 8–12.
- *The Graphics Screen and Accuracy* — information to help you understand the graphical and numerical limitations of a graphics calculator.
- *Coordinate Geometry on an EL-9650/9900* — basic commands and a variety of problems, suitable for Years 9 and 10.
- *Population Modelling* — a variety of problems from simple exponential growth to Leslie matrices and difference equations, covering Years 7–12.
- *Sequences and Series on an EL-9650/9900* — basic commands and a variety of problems, suitable for Years 10–12.
- *Matrices on an EL-9650/9900* — suitable for Years 11 and 12.
- *Complex Numbers on an EL-9650/9900* — suitable for Years 11 and 12.
- *Programming an EL-9650/9900* — suitable for teachers and keen students.
- *Introduction to Complex Numbers* — complex numbers from the beginning, covering the basic operations, but set in the context of complex numbers as a mathematical structure.

All the programs listed in these notes can be found at the above web site. You will need a PC-Link Kit CE-LK2 to copy these programs from your computer to your calculator.

1 Introduction

Graphics calculators lend themselves very nicely to demonstrating the visual aspects of Calculus — graphs of functions, tangent lines, areas under curves, etc — as well as to calculating numerically (approximating) many of the quantities that arise — derivatives and definite integrals, maximum and minimum values, etc.[1]

They can be used at a number of levels.

- As a basic graph plotter — what does the graph of $y = e^x$ look like?
- To investigate 'what if' questions, for example what happens if you change the parameters a and b in the equation $y = ae^{bx}$?
- To do (numerically) many of the basic calculations in Calculus, such as finding the slope at a point on a graph, definite integrals, maxima and minima, etc.
- To illustrate graphically, perhaps by way of a program, some of the concepts of Calculus. Two examples are showing how a secant line tends to a tangent line in the appropriate limit and how we can approximate the area under a graph by the areas of some rectangles. With sufficient ingenuity, almost anything can be done here, the only limitation being the small screen of the calculator.
- To automate, using the built-in operations or programs, some of the calculations that arise in Calculus: numerical integration methods, solving differential equations numerically, and so on.

At a more mundane level, graphics calculators are fun. Students pick up the operations very quickly (much faster than teachers), and if you can't get your students to use a graphics calculator, there are heaps of games to tempt them.

The other good news is that there are lots of resources available, many free on the Web, as well as a rapidly increasing number of books on using graphics calculators in almost every aspect of Mathematics and Science.

Getting started is always the hardest, especially when you have to modify or write new courses, but the experience at ADFA and most other schools and universities at which graphics calculators have been used for a while, is that graphics calculators should not just be an add-on to a course, but should be integrated fully, including their use in tests and exams. This raises many issues, most of which are resolvable. You might like to read for example *Graphics calculators in the mathematics curriculum: Integration or differentiation?* by Jen Bradley, Barry Kissane and Marian Kemp about their experiences in WA.[2]

At UNSW@ADFA, we have been using graphics calculators (TI-83+s at present) in our first-year courses since 1993 and have come to appreciate their worth in learning Mathematics.

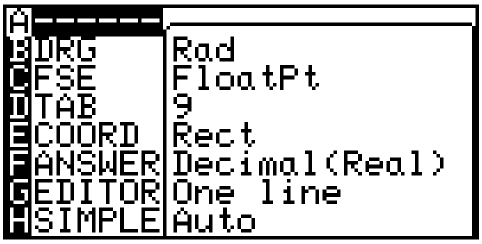
[1] There are, of course, 'calculators' that go even further and do things symbolically.

[2] This paper is available at *wwwstaff.murdoch.edu.au/∼kissane* under Publications. There are a number of other interesting papers here too.

2 Basic Calculus Operations

The key with +, −, ×, ÷ on it (first column, second row) we shall call the Home key — pressing it returns to the Home screen.

Before starting, press SET UP (2ndF BS) and set your calculator as shown.



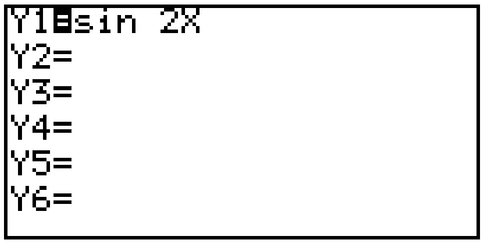
1. **Graph $f(x) = \sin(2x)$ for $0 < x < \pi$**

   Note that the graphics and table keys are the top row of keys.

   - Press **Y=**: set Y1 = sin 2X.

     The independent variable X is the X/θ/T/n key in the right-hand column of keys.

     Note the highlighted = sign, which means the function will be plotted when you press GRAPH. Move the cursor over the = sign and press ENTER to toggle the function off/on.



   - Press **WINDOW**: set the viewing window.

     For X, suitable values here are:
     Xmin = *0* *Xmax* = π *Xscl* = *0.5.*

     Just type in the value with the cursor at the start of each line — you don't have to move it to after the = sign. Press ENTER or the down arrow after each value, *including the last.*

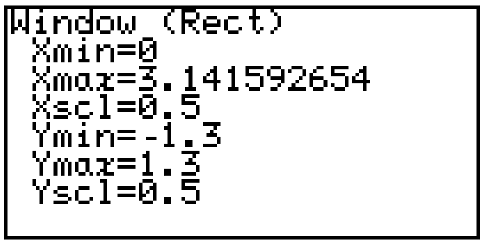
     π is 2ndF (−).

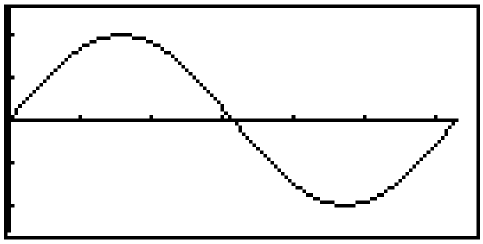
     Xscl is the distance between tick marks on the X axis (cosmetic only: 0 gives no tick marks).

     Suitable Y values here are:
     Ymin = −1.3 *Ymax* = *1.3* *Yscl* = *0.5.*

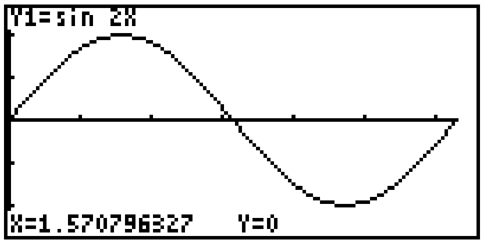
     Note the difference between the subtract key − and the change-sign key (−).

     You could also use ZOOM *Auto*, which calculates appropriate Y values for the function(s) and X values you set.

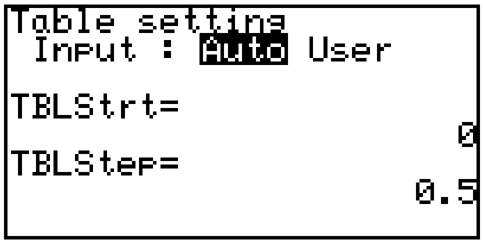




- Press **GRAPH**: graph the function.



- Press **TRACE**: move the cursor along the graph with the left- or right-arrow keys; the cursor coordinates are shown at the bottom.

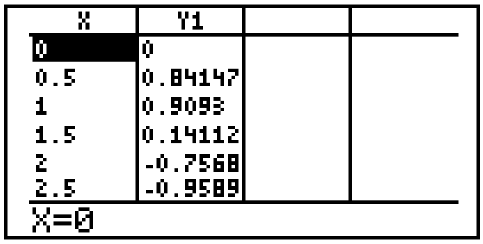


- **TABLE**

  Set the table 'WINDOW' as shown using TBLSET (2ndF TABLE).

  Press ENTER after each value.

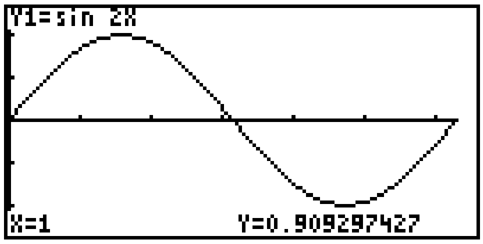
  Select *Auto* with the cursor and ENTER.



Press TABLE: move around the table with the arrow keys. Note that you can scroll up or down indefinitely.

2. Estimate $f(1)$

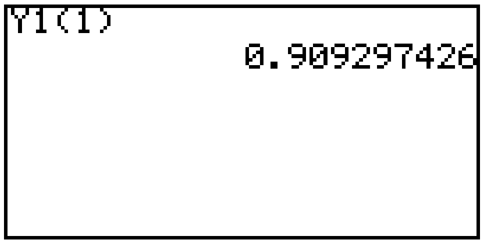
- **On the graph**
  - Press GRAPH (*top row of keys*).
  - Press CALC (2ndF TRACE) 1, type in the X value, 1 in this case, and then ENTER to move to the desired point on the graph. Note the coordinates at the bottom of the screen.

    Alternatively, press TRACE and use the left and right arrow keys to move the cursor along the curve (but note the problem that arises when trying to reach X=1). The up and down arrows move between functions if there is more than one graphed.

- **On the home screen**
  - Type in *Y1(1)* ENTER. *Y1* is VARS (2ndF X/θ/T/n) ENTER 1.

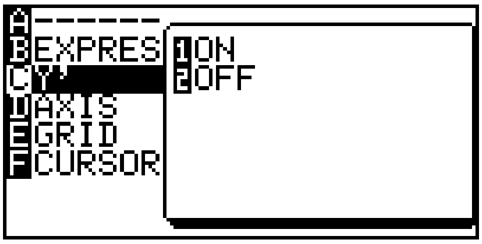
    You can't just type Y 1.

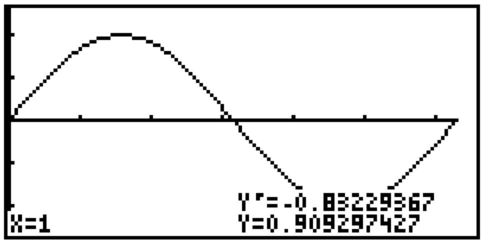
    

- *Answer*: $f(1) = 0.90930$, rounded to 5 decimal places.

3. Estimate $f'(1)$

- **On the graph**

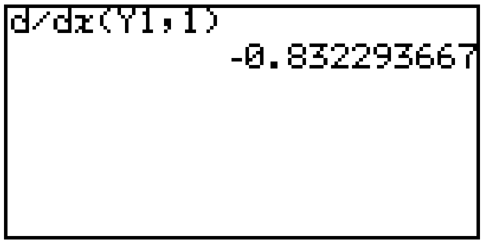




  - Press [FORMAT] ([2ndF] [ZOOM]) [C] (Y′) and select ON. Press [GRAPH] to return to the graph.
  - Select *Value* in the CALC menu ([2ndF] [TRACE]).

    Type in the X value: [1] [ENTER].

    Alternatively, use the arrow keys to move the cursor to a point on the curve.
  - *T_line* in the [DRAW] DRAW menu draws the tangent at the cursor.



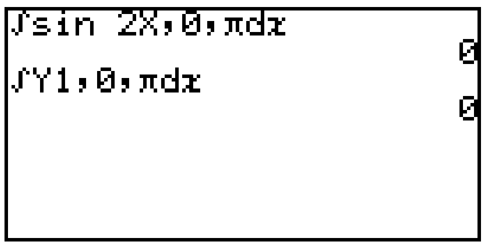
- **On the home screen**
  - *d/dx (Y1, 1)*.

    d/dx is in the [MATH] CALC menu.

    Y1 is [VARS] [ENTER] [1].

- *Answer*: $f'(1) = -0.83229$, rounded to 5 decimal places.
- *Accuracy*: can be adjusted in *d/dx* by an optional third argument. The default value for this argument is 10^{-5}.

4. Estimate $\int_0^{\pi} \sin(2x)\, dx$



- **On the home screen**
  - $\int$ *sin 2X, 0,* π *dx* or $\int$ *Y1, 0,* π *dx*.

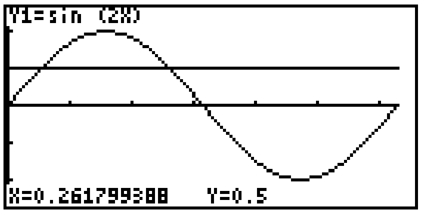
    $\int$ and *dx* are in the MATH CALC menu.

- *Answer*: $\int_0^{\pi} \sin(2x)\, dx = 0$.

  The accuracy can be specified by an optional third argument.

5. Solve $\sin(2x) = 0.5$ for $0 \leq x \leq \pi$

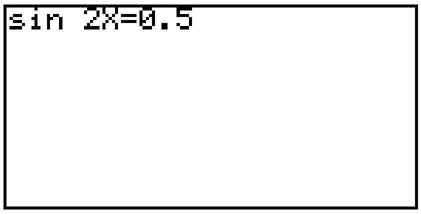
- **On the graph**
  - Graph Y1 and Y2 = 0.5.
  - Use *Intsct* in the CALC menu: this calculates the intersection point of one of the intersections on the screen. Keep choosing *Intsct* until it finds the one you want.



- **From the home screen**
  - Use the *Solver* (2ndF PRGM). Unless you have a very simple equation, choose *Newton* as your method.

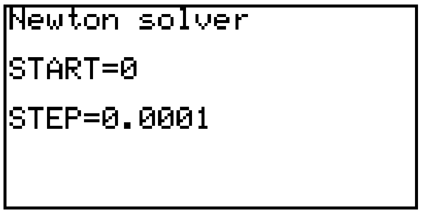
    Enter the equation as

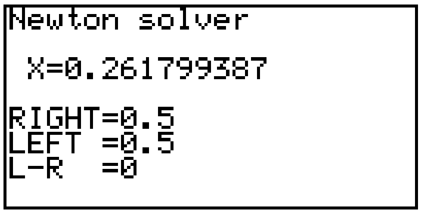
$$\sin\ 2X = 0.5.$$



    and press ENTER. = is ALPHA (–).

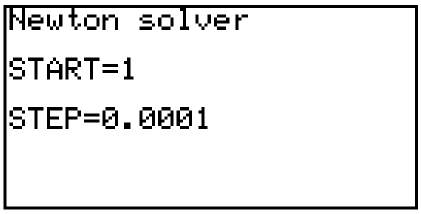
    Then press EXE (2ndF –). The calculator will display the START value (X) and the STEP size for its search. Change START to 0, press ENTER to record it and then press EXE to carry out the calculation.



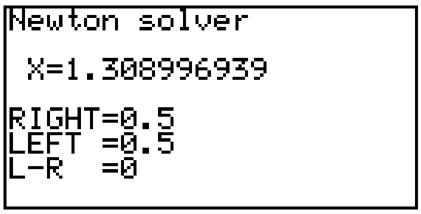


    If there is more than one solution or if you want greater accuracy, press CL and EXE: change START and/or STEP to the new values (pressing ENTER after each change), and press EXE again to carry out the calculation.

    Keep pressing CL if you want to change the equation.



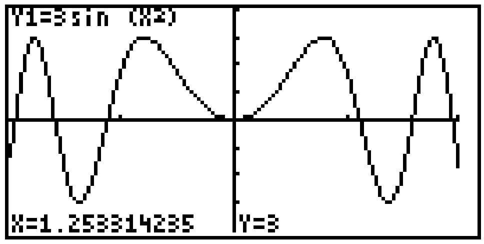
- *Answer*: the curves intersect at $x = 0.26180$ and $x = 1.30900$, both rounded to 5 decimal places.



There are some subtleties in the SOLVER that I have not appreciated yet. The manual may be of some use if you want to use this feature. I think the graphic approach is better, as you can see what you are trying to find.

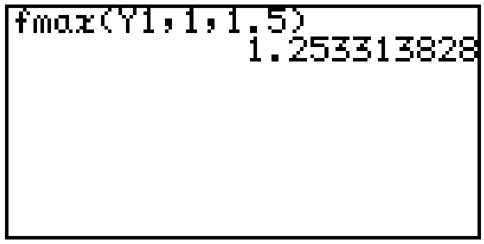
6. Find the value of the first positive maximum of $3\sin(x^2)$

- **On the graph**
  - Set $\text{Y1} = 3\sin \text{X}^2$.
    Turn off or clear all other functions.
    Set a WINDOW $[-\pi, \pi, \pi/2,] \times [-4, 4, 1]$ and press GRAPH.
  - Select *Maximum* in the CALC menu. This finds the first (local) maximum on the screen, starting from the left. Keep Selecting *Maximum* until you find the maximum you want.
  - Press the Home key and type 2X^2 ENTER on the home screen. You should recognise the first 6 digits of the number.



- **On the home screen**
  - *fmax (Y1, 1, 1.5)*.

    fmax is in the MATH CALC menu. The last two inputs are the bounds for the search.



- *Answer*: the maximum value of 3 occurs at $x = 1.25331$, rounded to 5 decimal places.

The CALC operations *Minimum*, *X_Incpt*, *Y_Incpt* and *Inflec* work in the same way as *Maximum*.

3 A Classic Problem

A hare and tortoise compete in a one-kilometre race. The distance each competitor has travelled from the starting point is given by a formula. In time t **minutes**, the distance in **metres** travelled by the hare is given by $H(t) = \frac{500}{3}(2\sqrt{t} + \sqrt[3]{t})$, while the distance in **metres** travelled by the tortoise is given by $T(t) = 100t + 250\sqrt{t}$.

Press [Y=] and enter the formulas for H and T in Y1 and Y2 respectively. You have to use X ([X/θ/T/n]) as the independent variable. The cube root is [3] [2ndF] [a^b].

Set your WINDOW so that the two graphs go from the bottom left to the top right of the screen. *Hints*: The race takes about 5 minutes. How far is the race?

The race looks better if the two graphs are plotted simultaneously. Ask your teacher how to do this.

Answer the following questions, writing down the steps you took. You will need to plot the lines Y3 = 500 and Y4 = 1000 and use *Intsct* in the CALC menu. You may need to increase *Ymax* when using *Intsct* so that you can see better the point you are interested in.

1. Who gets to the halfway point first? How long does it takes them? Verify your answer algebraically.

2. What is the time and distance at which the two runners are neck and neck?

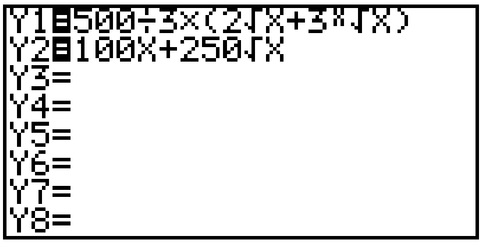
3. Who wins the race, by what time margin and by what distance margin?

The printed version of this activity contains a picture here. Unfortunately we cannot include it in the web version for copyright reasons. Unfortunately also, we can no longer find it on the web.

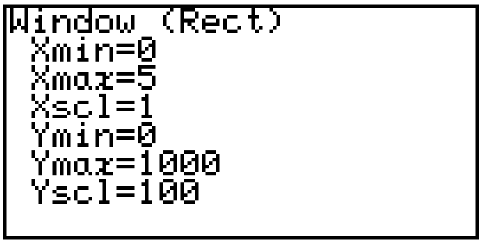
However, there is a suitable picture at *http://www.jimnuttle.com/illo1/torhar.gif*, which you might like to include if you download this activity from the web.

Teacher's Notes

The questions in this version have been written deliberately in general terms for a good class. For a less-advanced class, students may need to be led a little through each question. For example: *What equation do we need to solve to answer this question? What does this mean about the graphs of each side of the equation? How do we solve this equation on the calculator?* and so on.

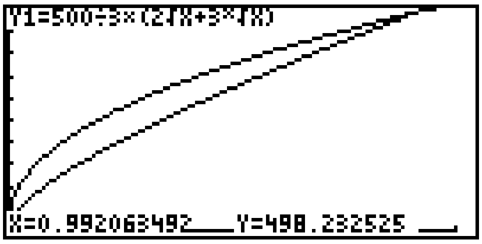


Press [Y=] and put the equation for the hare in Y1 and that for the tortoise in Y2. Be careful to put in the × before the bracket in Y1. You might like to discuss with the class how to write the formulas in a suitable form for the calculator. Time t becomes X on the calculator.

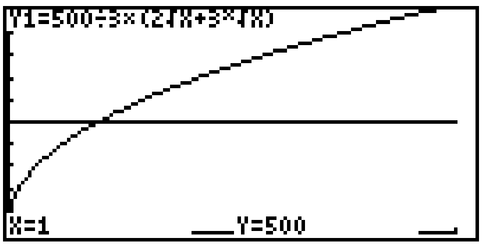


Then set the WINDOW. Discuss first what each axis represents and suitable scales. The Y axis represents distance in metres, so $0 < Y < 1000$. The winner is then the competitor whose graph first reaches the top of the screen (providing the graphs are plotted simultaneously — see below).

The time (X) scale has to be guessed. The race takes a little less than 5 minutes, so (after experimenting) $0 < X < 5$ gives a good view. Set Xscl, the distance between the tick marks on the X axis to 1 and Yscl to 100. If either of these is too small, you will get a double line for the axis.



1. Press [TRACE]; use the up/down arrows to see which graph is which — you need EXPRES ON in FORMAT ([2ndF] [ZOOM]) for this.
The hare clearly reaches the halfway point (500 m) first.



To find how long it took, we solve $H(t) = 500$ for t: set $Y3 = 500$ and find the intersection of Y1 and Y3 using *Intsct* in the CALC menu.

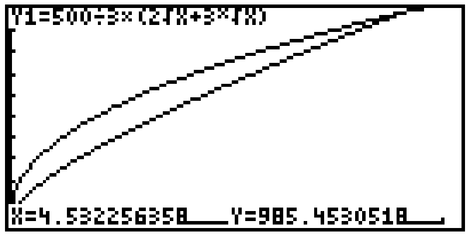
When you select *Intsct*, the calculator will find one of the intersections on screen. Keep selecting *Intsct* until you find the intersection you want. In the figure, Y2 is turned off.

The value for t is 1 minute, a value we can confirm algebraically to be exact by substituting $t = 1$ into the equation for the hare. Note that it is easy to **verify** that $t = 1$ is a solution, but tricky to **solve** $H(t) = 500$ algebraically (it turns into a cubic equation).

The hare reaches the halfway point first in a time of 1 minute.

2. To find when they are neck and neck, we have to solve $H(t) = T(t)$, that is find the intersection of Y1 and Y2 (algebraically, this turns into a quartic equation).

Because of the way *Intsct* works, it is probably a good idea to turn off any functions not involved (cursor on the = sign in [Y=] and press [ENTER]).

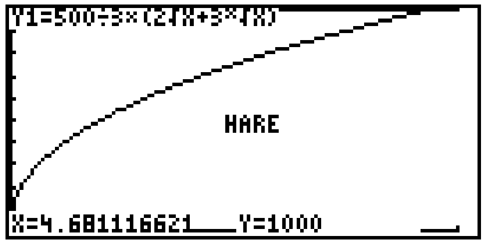


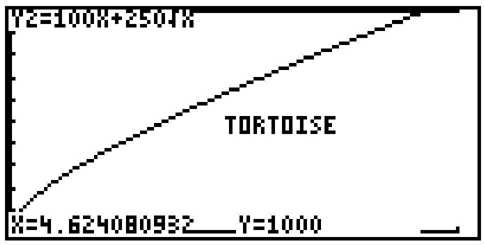
We obtain, using *Intsct* (twice), $t = 4.53$ minutes and distance equal to 985 m, both accurate to 3 significant digits. It might be useful to [ZOOM] *In* on this part of the graph to see the two curves more clearly.

The hare and tortoise are neck and neck after about 4.53 minutes or about 4 minutes 32 seconds, at a distance of about 985 metres from the start.

3. To find the winner, we have to determine the time at which each competitor reaches the finish (1000 m).

Setting Y4 = 1000, we find the hare finishes at $t = 4.681$ minutes (intersection of Y1 and Y4) and the tortoise finishes at $t = 4.624$ minutes (intersection of Y2 and Y4).





To find the distance margin, calculate $H(4.624)$, the position of the hare when the tortoise finishes: $H(4.624) \equiv \text{Y1}\,(4.624) = 994.45$ m, to 5 significant digits.

Y1 is in the [VARS] ([2ndF] [X/θ/T/n]) EQVARS XY menu.

The tortoise wins the race by a margin of 0.057 minutes or 3.42 seconds. The distance margin is 5.55 m.

Setting simultaneous graph plotting

The only way to set simultaneous graph plotting seems to be via a program. Press [PRGM], select NEW and call the program SIMUL. Press [PRGM] again, then [F] and scroll down to find *Simul*. Press [ENTER] to select it. This is your program.

Press the Home key (the one with +, −, × and ÷ on it), press [PRGM], select EXEC, scroll down to your program and press [ENTER] to run it. Graphs will now plot simultaneously.

To go back to sequential graphing, use the corresponding program SEQUEN, containing the single command *Sequen*, or reset defaults in [OPTION] [C].

Using the Newton-Raphson Method

The problems here are also a good application of the Newton-Raphson method for finding the zero of a function. For this, it is useful to have a program.

The program NEWTON for the EL-9650/9900 can be downloaded at *www.ma.adfa.edu.au* under *High School and College Activities*. The program finds the zero of a graphed function, given an initial guess provided by the position of the cursor on the graph.

The program finds zeroes of the function in Y1, so it is useful to put the hare and tortoise equations in Y2 and Y3. Y1 can then be defined as Y1 = Y2 − 500 to find when the hare reaches halfway; Y1 = Y2 − Y3 to find when they are neck and neck; Y1 = Y2 − 1000 to find when the hare finishes and Y1 = Y3 − 1000 to find when the tortoise finishes.

4 Maximum and Minimum

When manufacturers are designing their packaging, they must keep in mind the amount of product that has to fit inside and the amount of material it will take to make the package. Consider the humble soft-drink can. The standard volume is 375 mL or 375 cm^3. Any number of cans can be designed that will hold this volume of liquid, but they will vary in shape and therefore in the amount of material needed to make the can (and therefore cost).

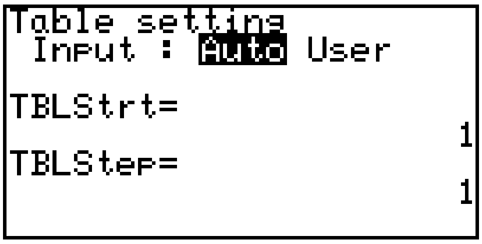
The formula for the volume of a cylinder V in terms of radius r and height h, is

$$V = \pi r^2 h.$$

Rearrange the volume formula to make h the subject and let the volume be 375 cm^3. *What are the units of r and h?*

$h =$

Enter this formula as Y1, with X as radius r.
X is the [X/θ/T/n] key.
To check, enter the volume formula: Y2 = π X^2 Y1.
Y1 is in the [VARS] ([2ndF] [X/θ/T/n]) EQVARS menu.



Press [TBLSET] ([2ndF] [TABLE]).
Set *TBLStrt* = 1 and *TBLStep* = 1, pressing [ENTER] after each value.
Press [TABLE].
Do you get the correct value for the volume in Y2?

Write down the formula for the surface area of a cylinder, including the ends.
The surface area determines the amount of material needed to make the can. *Why?* Enter the formula for surface area in Y2 in terms of X (radius) and Y1 (height).

SA =

Y2 =

View the table of values again. *What do you notice about the values of the surface area?*

Use [TBLSET] to set new starting values and smaller steps to find the minimum surface area and corresponding radius (radius accurate to 1 decimal place).

Minimum SA =

Radius =

Now graph the surface area as a function of radius, and use *Minimum* in the CALC menu to find the minimum.

Draw your graph here.

Write down your values for the radius, height, surface area and circumference of the can when the surface area is a minimum. *Do these values seem reasonable?*

How does this compare with a can of soft drink? Why the differences? What about other cans?

You might like to read the article *The Best Shape for a Tin Can* by P. L. Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993).

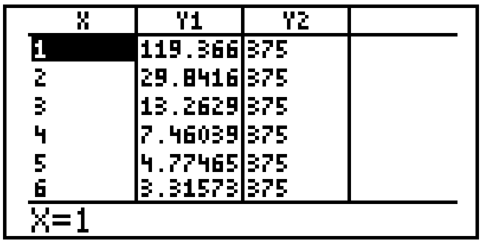
Teacher's Notes

This is an activity from: *Integrating the Graphics Calculator into Year 9 and Year 10 of the Victorian Mathematics CSF*, Teachers Teaching with Technology (T^3), 1998, modified somewhat. There are a number of maximum/minimum activities for all years that work in the same way.

The height of the cylinder is given by $Y1 = 375/\pi X^2$, where X is the radius r.

As a check, enter the volume formula $Y2 = \pi X^2 Y1$.

Press TABLE.



X	Y1	Y2
1	119.366	375
2	29.8416	375
3	13.2629	375
4	7.46039	375
5	4.77465	375
6	3.31573	375

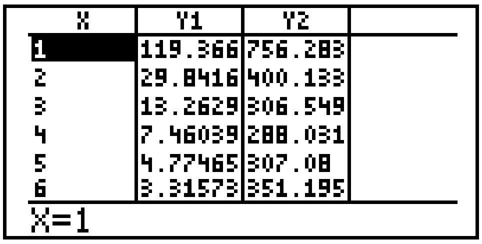
X=1

The total volume of the metal used to make the can, assuming the walls are of uniform thickness, is just the surface area times the thickness. Minimum surface area therefore means minimum volume of metal.

The surface area of a cylinder, including the ends, is given by

$$SA = 2\pi r^2 + 2\pi rh = 2\pi r(r + h).$$

View the table of values again. What do you notice about the values of the surface area?
The surface area decreases then increases as the radius increases. There is a (local) minimum.

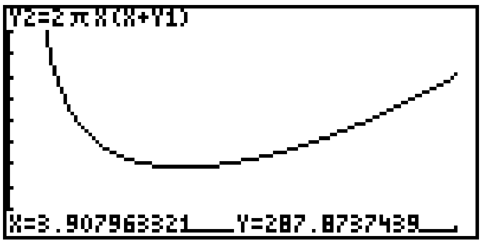


X	Y1	Y2
1	119.366	756.283
2	29.8416	400.133
3	13.2629	306.549
4	7.46039	288.031
5	4.77465	307.08
6	3.31573	351.195

X=1

With *TBLStrt* = 3 and *TBLStep* = 0.1, we find a radius of 3.9 cm for minimum surface area.

Alternatively, one can view the graph of surface area versus radius and use *Minimum* in the CALC menu to find the minimum. The figure has a WINDOW of $[0, 10, 2] \times [0, 1000, 100]$.



Again, we obtain a value of $r = 3.9$ cm (accurate to 1 decimal place) for the radius giving minimum surface area.

If your students have sufficiently developed calculus skills, they could prove algebraically that the global minimum lies at $r = \sqrt[3]{375/2\pi} \approx 3.9$. More generally, for a given volume V, it is not too hard to show that $h = 2r$ (height = diameter) for the least surface area.

Is this a reasonable answer for the radius?

Other factors may have to be taken into account such as what circumference is comfortable for the average human hand, the wastage of material when cutting the ends and the cost of making the joins.

radius $r = \sqrt[3]{\frac{375}{2\pi}} \approx 3.9\,\text{cm}$ height $h = \sqrt[3]{\frac{1500}{\pi}} \approx 7.8\,\text{cm}$ ratio $\frac{h}{r} = 2$

surface area $\approx 288\,\text{cm}^2$ circumference $\approx 24.6\,\text{cm}$

How does this compare with a can of soft drink?

A soft-drink can has a radius of 3.25 cm and a height of 13 cm: $h/r = 4$. Its surface area is about $332\,\text{cm}^2$ and circumference 20.4 cm. The article *The Best Shape for a Tin Can*[3] by P. L. Roe, either in *The Mathematical Gazette* 75, 472 (1991) or reprinted in *The College Mathematics Journal* 24, 233 (1993) goes into why there might be differences between the theory here and the actual values. A good example of mathematical modelling.

[3]A copy of this article is available from Peter McIntyre.

5 Discovering the Derivative by Exploration

Modified from: Judy Broadwin, *Discovering the derivative by exploration*, TI-82/83 Activities for Calculus, TI Web site.

We will put several different functions $f(x)$ in Y1 and try to discover their derivatives using the difference quotient

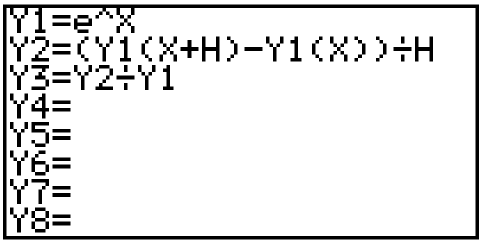
$$DQ = \frac{f(x+h) - f(x)}{h},$$

where h is a small number. Store 0.0001 in memory H: 0.0001 [STO] [ALPHA] [H] [ENTER].

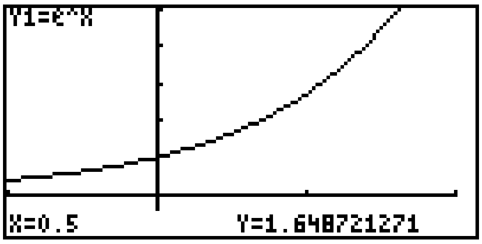
Enter the difference quotient into Y2: Y2 = (Y1(X + H) − Y1(X))/H.
Y1 is in the [VARS] EQVARS menu. You can't just type [Y] [1].

Turn off Y2: move the cursor over the = sign and press [ENTER].

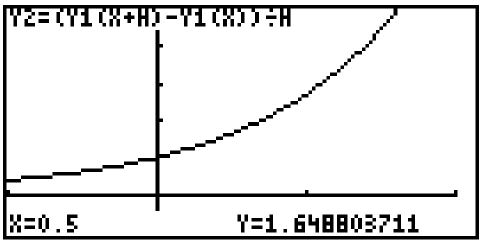
Exploration 1 *Discover the derivative of $f(x) = e^x$*



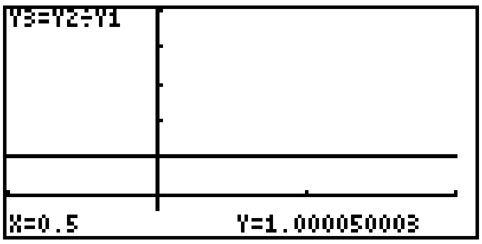
Set Y1 = e^x.
Set [WINDOW] to $[-1, 2, 1] \times [-1, 5, 1]$.
Press [TRACE].



Then turn off Y1 and graph Y2. *What do you see?*



Then set Y3 = Y2/Y1 and graph it. *What do you see?*
What is $\frac{d}{dx}(e^x)$*?*

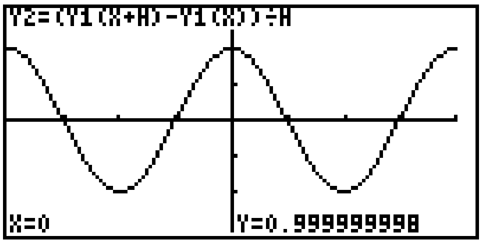


Exploration 2 *Discover the derivative of* $f(x) = \sin(x)$

```
Y1=sin X
Y2=(Y1(X+H)-Y1(X))÷H
Y3=cos X
Y4=
Y5=
Y6=
Y7=
Y8=
```

Set Y1 = sin X.
Turn Y1 off and Y2 on.

Press [ZOOM] [E] [1] and then [TRACE].
Describe the graph of Y2.
Do you recognise the function?



Can you describe the relationship between the graph of the function and the graph of its difference quotient?

Set Y3 = cos X.
In [TBLSET], set *TBLStrt* = 0, *TBLStep* = 1.
Compare Y2 and Y3 using [TABLE].
Why are they not identical?

X	Y2	Y3
0	1	1
1	0.54026	0.5403
2	-0.4162	-0.4161
3	-0.99	-0.99
4	-0.6536	-0.6536
5	0.28371	0.28366

X=0

Exploration 3 *Discover the derivative of* $f(x) = \ln(x)$

Set Y1 = ln(X) and turn it off. Leave Y2 on.

In [TBLSET], set *TBLStrt* = 0, *TBLStep* = 1.
Press [TABLE].
Can you guess the derivative of $\ln(x)$*?*

X	Y2
0	-------
1	0.99995
2	0.49999
3	0.33333
4	0.25
5	0.2

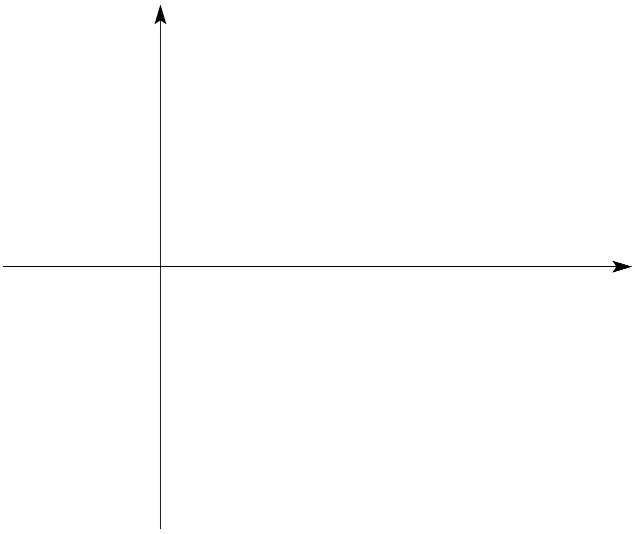
X=1

6 Derivatives and the Shape of a Graph

From: *This is I.T. Graphics-calculator activities for upper-secondary students* by Pat Forster, Alan Cadby and Gary Young, AAMT, 1998.

This investigation is to help you understand how the first and second derivatives of an equation can tell you the shape of its graph. You will also learn some new terminology to describe graphs.

1. Copy from your calculator the graph of $y = 5x^2 - 2x^5$ for $-1 \le x \le 2$, $-10 < y < 10$. Put scales on your graph.



2. Find and enter the equations for the first and second derivatives of $y = 5x^2 - 2x^5$ into your calculator as Y2 and Y3. Look at the values of these derivatives for different values of x using the calculator's table of values as follows:
first press TBLSET and set *TBLStrt* $= -1$; *TBLStep* $= 0.2$. Then press TABLE.

(a) Use a red pen and put plus signs (+) along the sections of your graph above where dy/dx is positive. Similarly, put minus signs (−) and zero (0), where appropriate.

(b) Using a different-coloured pen, mark where d^2y/dx^2 is +, − or zero. (Scroll up the table in the X column.)

(c) The points on a graph where $dy/dx = 0$ are called **stationary points**. Fill in the table below.

Type of stationary point	Co-ordinates	$\frac{dy}{dx}$ (+, −, 0)	$\frac{d^2y}{dx^2}$ (+, −, 0)
maximum			
minimum			

3. (a) Sections of a graph where $\dfrac{d^2y}{dx^2}$ is positive are described as **concave up**.
If $\dfrac{d^2y}{dx^2}$ is negative, the curve is **concave down**.

For what values of x is your graph concave up? ....

For what values of x is your graph concave down? ....

Is the graph at the maximum point concave up or concave down? ....

Is the graph at the minimum point concave up or concave down? ....

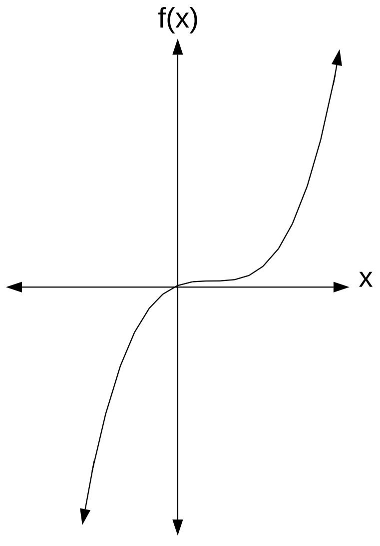
(b) A point at which a curve is concave down (d^2y/dx^2 negative) on one side and concave up (d^2y/dx^2 positive) on the other side, *or vice versa*, is called a **point of inflection**. The sign of the second derivative must *change* as we pass through a point of inflection because the graph changes concavity.

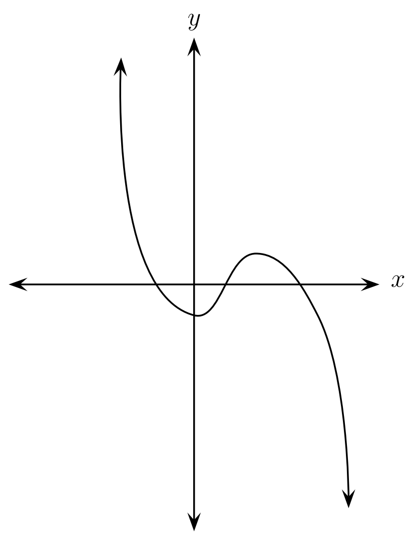
Use the table of values to find the point of inflection of the function here: change *TBLStrt* and *TBLStep* to zoom in on the zero of Y3. *Why do we look for a zero? Is this zero a point of inflection?* Write its co-ordinates on your graph.

(c) *Is $x = 0$ a point of inflection of the function $y = x^4$?* Give reasons.

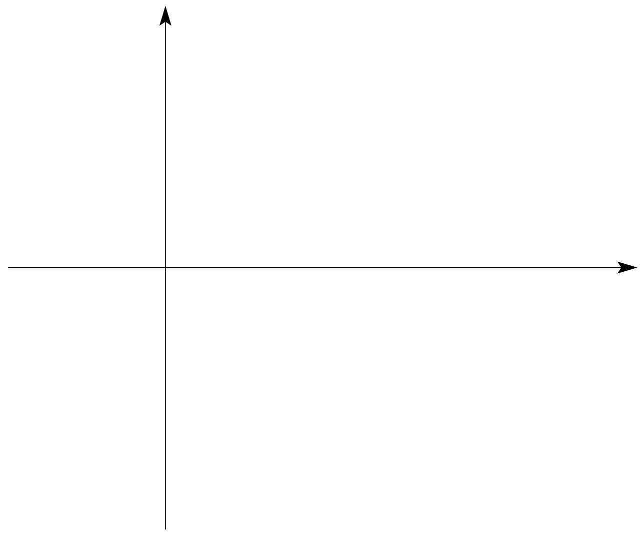
(d) Points of inflection can be the steepest section of a graph — look at the graph of $y = \sqrt[3]{x-1}$ at $x = 1$. They can also be stationary points (graph is horizontal) — look at the graph of $y = x^3 + 1$ at $x = 0$. However, in general, the slope at a point of inflection can be any value.

Put a cross on the points of inflection in each of the following two graphs. *Is the point of inflection in either graph a stationary point?*





4. Sketch a graph with the given properties.



(a) Endpoints $(-2, 1)$ and (7,6).

(b) Stationary point at $(1, -5)$ where $\dfrac{d^2y}{dx^2}$ is positive.

(c) Stationary point at (3,2) where $\dfrac{d^2y}{dx^2} = 0$.

(d) $\dfrac{d^2y}{dx^2} = 0$ at $(2, -1)$, $\dfrac{d^2y}{dx^2}$ is negative between $x = 2$ and $x = 3$, and $\dfrac{d^2y}{dx^2}$ is positive for $x > 3$.

(e) Classify the points in (b) – (d) as a maximum, minimum or point of inflection.

....

(f) For what values of x is the curve concave up?

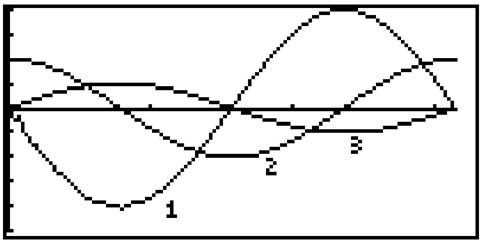
....

7 Graphing Derivatives and Anti-derivatives

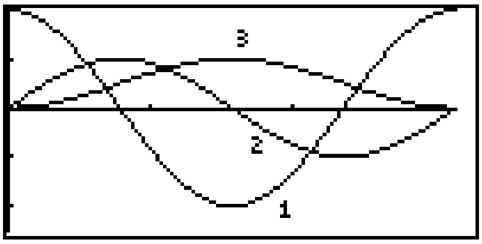
Based on: *How do you graph derivatives and anti-derivatives?* by John Maloney, Eightysomething! 7(1), 1997.

Here are two questions to challenge your students' understanding of the concepts of derivative and anti-derivative.

Question 1: The figure plots a function, its derivative and its second derivative. *Which curve is which?*

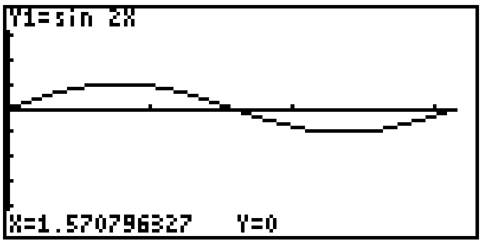


Question 2: The figure below plots a function, its derivative and an anti-derivative. *Which curve is which?*



Let's investigate how to graph a function, its derivative and anti-derivative. Press SET UP and make sure that *Rad* (radians) is selected.

Press Y= and set Y1 = sin 2X and set a WINDOW of $[0, \pi, 1] \times [-5, 4, 1]$. Press TRACE.

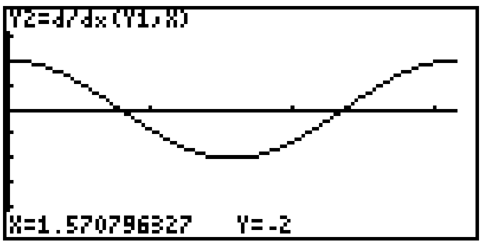


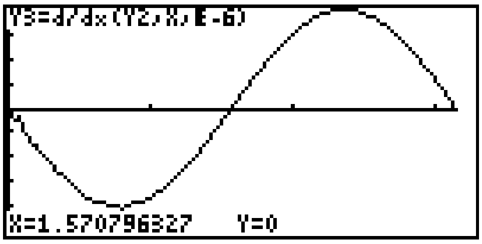
Plotting derivatives

To graph the first derivative, set Y2 = d/dx (Y1, X). *d/dx* is MATH CALC 0 5.

Y1 (in the VARS EQVARS menu) is the function we are differentiating and X is the value at which we wish to calculate the derivative — this value is set by the grapher as it plots successive points on the graph.

Pressing TRACE gives you the figure.

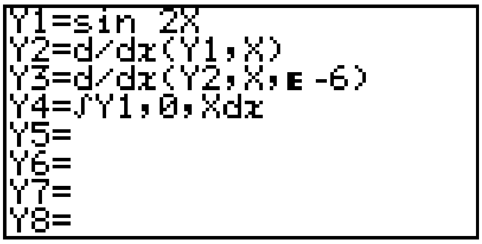




Set Y3 = d/dx (Y2, X, E−6) to graph the second derivative, the derivative of the first derivative.
The E−6 (Exp (–) 6) — tolerance of 10^{-6} — is to improve the accuracy to eliminate some numerical instability at small X with the default tolerance of 10^{-5}.

Note: The EL9650/9900 will allow you to calculate the third derivative in this manner, but it will not be very accurate. You need to use a different difference approximation[4] to the central or symmetric difference approximation used in *d/dx*.

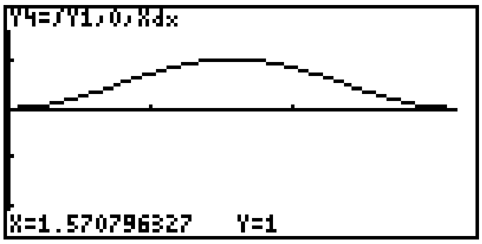
With Y1, Y2 and Y3 turned on, you should obtain the graph of the first question above. Students, of course, must use the relationship between a function and its derivative to answer the question.



Plotting an anti-derivative

To plot an anti-derivative of Y1, set Y4 = $\int$ Y1, 0, X *dx*. *int* and *dx* are in the MATH CALC menu. Y1 is the function we are integrating,[5] and 0 and X are the integration limits.

The function we are plotting is therefore $y(x) = \displaystyle\int_0^x \sin(2t)\, dt$.



Turn off Y1 – Y3, set *Ymin* = −2.5, *Ymax* = 2 and press TRACE.

Note: The lower integration limit 0 is arbitrary. Change it to other numbers to show students that a function has (infinitely) many (related) anti-derivatives. The one graphed here is the one that passes through the origin.

With Y1, Y2 and Y4 turned on, you should obtain the graph of the second question above.

[4] $f'''(x) \approx \dfrac{f(x+2\Delta x) - 2f(x+\Delta x) + 2f(x-\Delta x) - f(x-2\Delta x)}{2(\Delta x)^3}$.

[5] We could have put the function in directly as sin 2X. Similarly when using *d/dx*. The advantage of the way we have set up the functions Y2 − Y4 is that we can see the derivatives and anti-derivative of whatever function is in Y1.

The Chain Rule

Set

$$\mathrm{Y5} = \mathrm{X}^2$$

$$\mathrm{Y6} = \mathrm{Y1}\,(\mathrm{Y5})$$

$$\mathrm{Y7} = \mathrm{d/dx}\,(\mathrm{Y1}, \mathrm{Y5}) \times \mathrm{d/dx}\,(\mathrm{Y5}, \mathrm{X})$$

$$\mathrm{Y8} = \mathrm{d/dx}\,(\mathrm{Y6}, \mathrm{X}).$$

Plot and compare the graphs of Y7 and Y8 using a WINDOW of $[-3, 3, 1] \times [-12, 12, 4]$. Use TRACE and the up/down arrows to identify the two curves. Explain what you see.

Answers to the Questions

Question 1 *One possible answer*

Curve 3 is initially positive, so that it cannot be the derivative of either curve 1 or curve 2, both of which have negative slopes initially. Therefore, curve 3 must be the function.

The initial slope of curve 3 is positive, so that curve 1, which is initially negative, cannot be the first derivative of curve 3. Therefore, curve 2, which is initially positive, must be the first derivative of curve 3.

Curve 1, initially negative, must be the derivative of curve 2, and therefore the second derivative of curve 3.

Question 2 *One possible answer*

Curve 3 cannot be the derivative of either curve 1 or curve 2 because it does not pass through zero when either of these curves has a maximum/minimum. Therefore, curve 3 must be either the function or an anti-derivative.

If curve 3 were the function, curve 2 could be its derivative (although its behaviour at either end doesn't look correct), but curve 1 could not be an anti-derivative because it is decreasing initially, whereas the anti-derivative of (positive) curve 3 must be increasing. Therefore, curve 2 must be the function and curve 3 an anti-derivative.

The anti-derivative of curve 2 must increase where curve 2 is positive and decrease where curve 2 is negative. Curve 3 has these properties. Because the area of curve 2 below the x axis is equal to the area above the x axis, the final y value of curve 3 must be equal to its initial y value.

Curve 1 must therefore be the derivative of curve 2. *Check*: when curve 2 has a maximum or minimum, curve 1 is zero; where curve 2 is increasing, curve 1 is positive, where curve 2 is decreasing, curve 1 is negative.

8 Rectangles, Area and the Definite Integral

The RIEMANN program illustrates graphically how the area under a graph can be approximated by the areas of rectangles. As the number of rectangles covering the area increases, we obtain a better approximation to the area. Here we approximate $\int_0^1 e^x\,dx$ by drawing rectangles (the sum of the areas is a *Riemann sum*).

- Put the function $f(x) = e^x$ in Y1.
- Set a WINDOW of $[0, 1, 0.2] \times [-0.3, 3, 1]$. The value of -0.3 for $Ymin$ allows room at the bottom of the screen for displaying the cursor coordinates.
- Run the program: press PRGM and select RIEMANN.
- Set the integration limits A = 0 and B = 1.
- Set the number of rectangles $N = 5$.
- Press 1 to select the Left-Endpoint Rule (LER). The program will plot the function and draw in 5 rectangles, each rectangle touching the curve at its top-left corner. In this case, the area of the rectangles clearly underestimates the area under the graph.
- Press ENTER to see the area of the rectangles (LHSUM) as an approximation to the area under the graph.
- Press ENTER and set $N = 5$ again, but choose the Right-Endpoint Rule (RER). This time we obtain an overestimate of the area under the curve.
- Repeat the above steps, doubling the number of rectangles each time. Fill in the table below.[6] Your best estimate: $\int_0^1 e^x\,dx \approx$ ________.

N	LER	RER	MEAN
5			
10			
20			
40			
80			

- When you've had enough, press ON CL.

[6] The mean of the two estimates is equivalent to the Trapezoidal-Rule approximation to the area, a more accurate approximation for a given N than either the Left- or Right-Endpoint Rules.

Answers *rounded to three decimal places.*

N	LER	RER	MEAN
5	1.552	1.896	1.724
10	1.664	1.806	1.720
20	1.676	1.762	1.719
40	1.697	1.740	1.718
80	1.708	1.729	1.718

Best estimate is 1.718.

The exact answer is $e - 1 = 1.718$ to three decimal places.

9 Approximating Definite Integrals

Modified from an ADFA Lab, which is itself based on a lab in *Resources for Calculus Collection, Volume 1: Learning by Discovery*, Anita Solow, editor, Mathematical Association of America Note 26, 1993.

In this lab, we shall be comparing several numerical approximations to

$$\int_0^1 (5x^4 - 3x^2 + 1)\,dx$$

with the exact answer obtained by algebraic integration. This will give us a feel for some of the methods of numerical integration, which we can then use for any function, including those which cannot be integrated algebraically.

Question 1 *Algebraic integration — the exact answer*

What is the exact value of this integral? You may not realise it, but you are using the *Fundamental Theorem of Calculus* to do this definite integral exactly.

Question 2 *Left-Endpoint and Right-Endpoint Rules*

One approach to numerical integration is to approximate the definite integral of $y = f(x)$ with $a \leq x \leq b$ by the areas of a number of rectangles under the curve. If a left-hand corner of each rectangle touches the curve, we have the *Left-Endpoint Rule*; if a right-hand corner of each rectangle touches the curve, we have the *Right-Endpoint Rule*. As the number of rectangles in the interval $[a, b]$ gets larger and larger (covering the integration range $a \leq x \leq b$ with more and more rectangles), both Rules give numbers closer and closer to the definite integral (exact answer).

(a) On Figure 1 (at the end of this Lab), sketch and shade in the rectangles for the Left-Endpoint-Rule approximation to the definite integral $\int_a^b f(x)\,dx$ with $n = 4$ (four rectangles). *Note*: The function in Figure 1 is not the function in Question 1.

(b) Using your sketch in (a), explain why the Left-Endpoint Rule with four rectangles approximates the area under the graph as

$$hf(x_0) + hf(x_1) + hf(x_2) + hf(x_3),$$

where $x_0 = a$, $x_4 = b$ and the width of each rectangle is $h = (b - a)4$.

(c) Use the RIEMANN program (instructions over) to estimate $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using the Left-Endpoint Rule with $n = 4$. A suitable WINDOW is $[0, 1, 0.1] \times [-0.3, 3, 0.5]$. Note that the integrand here is positive, so that the definite integral corresponds to the area under the graph of f.

(d) Now use the RIEMANN program, doubling n, the number of rectangles, until two successive answers are the same when rounded to two decimal places. Write down the n value of the first of these two answers.

Question 3 *The Trapezoidal Rule*

The Left-Endpoint and Right-Endpoint Rules approximate the area under a function by rectangles. In many cases, for example the function in Figure 1 with the rectangles you have drawn in, this is not a good approximation. We get a better approximation by using trapeziums: both top corners of each trapezium touch the curve.

(a) On Figure 2, draw in and shade the trapeziums, the total area of which approximates the definite integral $\int_a^b f(x)\,dx$, again with $n = 4$.

The area of the trapezium in Figure 3 is $h(r+s)/2$. To see this result, split the trapezium into two regions — a triangle and a rectangle.

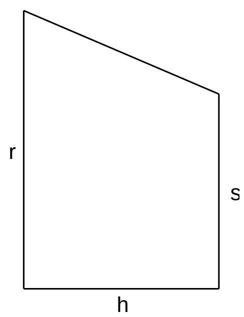


Figure 3

(b) Using your sketch in (a) and Figure 3, explain why the Trapezoidal Rule with four trapeziums approximates the area under the graph as

$$hf(x_0) + 2hf(x_1) + 2hf(x_2) + hf(x_3),$$

where $x_0 = a$, $x_4 = b$ and the width of each trapezium is $h = (b-a)/4$.

(c) Evaluate T_4, the Trapezoidal Rule with four trapeziums, as an estimate of the integral $\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ using the NUMINT program (T stands for Trapezoidal Rule). How does this result compare with the rectangle result and the exact answer?

(d) Now use the NUMINT program, doubling n, the number of trapeziums, until two successive answers are the same when rounded to two decimal places. Write down the n value of the first of these two answers. Compare it with the rectangle n value.

Question 4 *Simpson's Rule*

A picture of Simpson's Rule for which $n = 4$ is given in Figure 3. We want to estimate the area under the solid curve. We do this by fitting parabolas to three successive points on the graph. The dashed line shows two parabolas: one through $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$, the other through $\big(x_2, f(x_2)\big)$, $\big(x_3, f(x_3)\big)$ and $\big(x_4, f(x_4)\big)$.

(a) On Figure 3, shade the area calculated by Simpson's Rule as an approximation to the definite integral $\displaystyle\int_a^b f(x)\,dx$.

(b) Evaluate S_4, Simpson's Rule with four divisions of the integration interval, as an estimate of $\displaystyle\int_0^1 (5x^4 - 3x^2 + 1)\,dx$ (Simpson's Rule) using NUMINT. Note that the NUMINT version of Simpson's Rule uses 2N sub-divisions, where N is the value you input (n, the number of divisions of the integration interval, must be even for Simpson's Rule). Compare your result with those from Questions 1 – 3.

(c) Now use the NUMINT program, doubling n, the number of divisions of the integration interval, until two successive answers are the same when rounded to two decimal places. Write down the n value of the first of these two answers. Compare it with the rectangle and trapezium values.

Question 5 *Comparing the methods*

What conclusions can you draw from your results regarding the different methods for estimating the definite integral? Which method would you choose to use? Why?

Programs

These programs calculate approximate values for $\displaystyle\int_A^B f(X)\;dX$. The number N is an input to the program.

NUMINT approximates the integral using the *Left-Endpoint Rule* (**L**), the *Right-Endpoint Rule* (**R**), the *Trapezoidal Rule* (**T**) and the *Midpoint Rule* (**M**), **all with N sub-divisions**, and *Simpson's Rule* (**S**) **with 2N sub-divisions** to ensure an even number of sub-divisions.

RIEMANN approximates the integral using the *Left-Endpoint Rule* or the *Right-Endpoint Rule* with N sub-divisions of the interval [A, B], and draws the corresponding rectangles.

Use: Type the function to be integrated into Y1. [ON] [CL] stops both programs.

- For NUMINT, run the program and follow the prompts. Press [ENTER] to input a different number N of sub-divisions.
- For RIEMANN, first set a suitable WINDOW to display the function (plot the function on the integration range first to check). Run the program and follow the prompts. Make sure B > A, otherwise things get mixed up. Press [ENTER] after the graph is plotted to see the numerical approximation to the integral, and [ENTER] again to do a new plot.

Numerical Integration Lab Figures

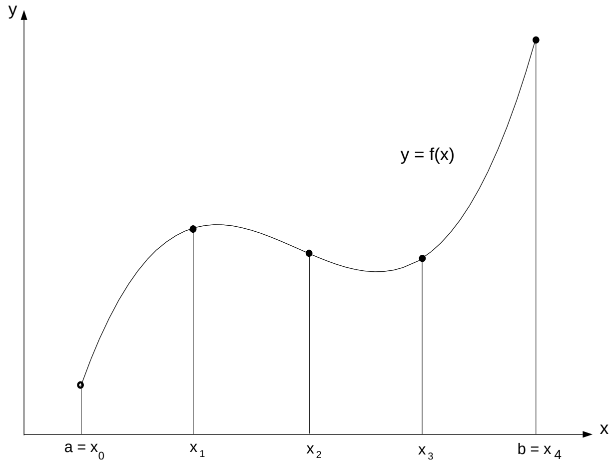


Figure 1: Left-Endpoint Rule

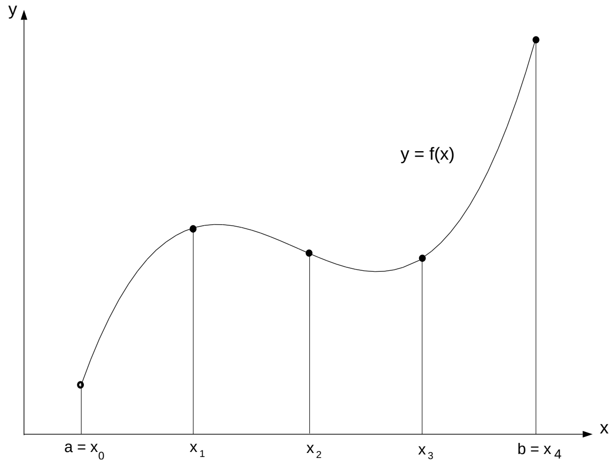


Figure 2: Trapezoidal Rule

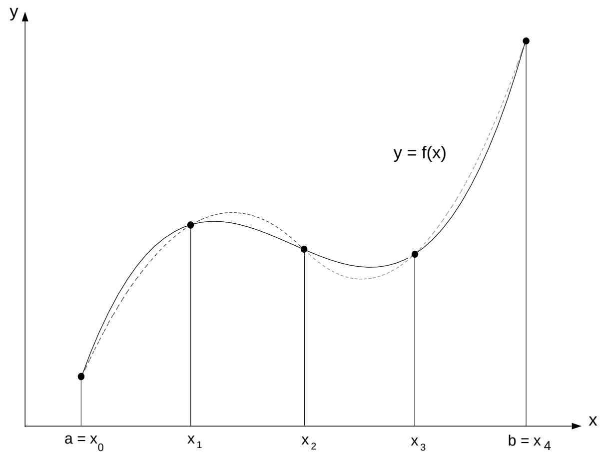


Figure 4: Simpson's Rule

10 Web Sites

- A good starting point for web links is the AAMT home page *www.aamt.edu.au*, in particular the links to other Maths sites under *Resources / Members' Sites / Mathematical Software and Technology*.

- Barry Kissane's home page *wwwstaff.murdoch.edu.au/∼kissane* is another web site of general interest. This page contains some interesting discussion papers under Publications and lots of other useful stuff, including resources.

- *www.prenhall.com/divisions/esm/app/calculator* contains instructions on how to do most things with most models of graphics calculator. A terrific reference site for basic calculator operations.

- *sharp-world.com/products/calculator/education/index.html*, the Sharp education site. Some useful materials here, including *Handbook Volume 1: Algebra* and an extensive set of classroom lessons using the EL-9600 (also applies to EL-9650/9900) by David Lawrence, both free to download.