Matrices on an EL-9650/9900

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At *www.unsw.adfa.edu.au/pems/news/high_school/hsc_activities.html*

- A variety of graphics-calculator activities for Years 9 and 10 — written as part of the CQTP Program for Sharp, Casio and TI calculators.
- *Using the EL-9650/9900* — an introduction to the basic operations, suitable for Years 8–12.
- *The Graphics Screen and Accuracy* — information to help you understand the graphical and numerical limitations of a graphics calculator.
- *Coordinate Geometry on an EL-9650/9900* — basic commands and a variety of problems, suitable for Years 9 and 10.
- *Population Modelling* — a variety of problems from simple exponential growth to Leslie matrices and difference equations, covering Years 7–12.
- *Sequences and Series on an EL-9650/9900* — basic commands and a variety of problems, suitable for Years 10–12.
- *Calculus on an EL-9650/9900* — suitable for Years 11 and 12.
- *Complex Numbers on an EL-9650/9900* — suitable for Years 11 and 12.
- *Programming an EL-9650/9900* — suitable for teachers and keen students.
- *Introduction to Complex Numbers* — complex numbers from the beginning, covering the basic operations, but set in the context of complex numbers as a mathematical structure.

All the programs listed in these notes can be found at the above web site. You will need a PC-Link Kit CE-LK2 to copy these programs from your computer to your calculator.

1 The Basics

In what follows, we refer to the key with $+$, $-$, $\times$ and $\div$ on it (first column, second row) as the *Home* key. Pressing this returns you to the Home screen.

The basic matrix operations work just like the operations with numbers.

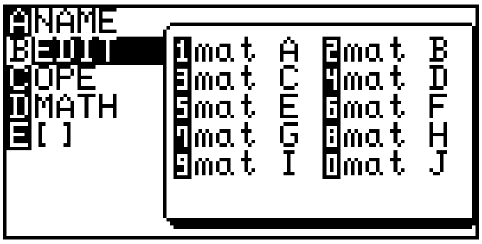
Let's begin by working out the matrix product $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Call the first matrix $\boldsymbol{A}$, the second $\boldsymbol{B}$.

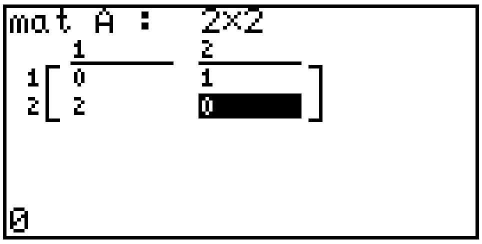
1.1 Putting matrices in your calculator

Press [MATRIX] ([2ndF] [STAT]) [B] (EDIT). You are now ready to edit one of the ten matrices mat A – mat J.

Press [1] to select mat A.



Now input the order (rows $\times$ columns) and the elements of mat A — the cursor indicates what is required. Press the desired number and [ENTER]. The cursor moves on and leads you through the elements row by row. Make sure you press [ENTER] after the last element.

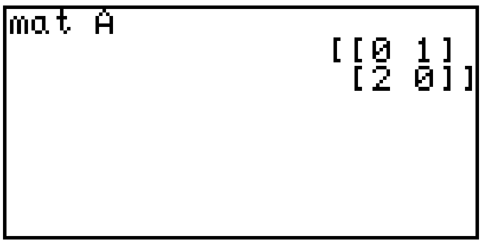


Press [MATRIX] again, but press [2] this time to input mat B.

Press the Home key to return to the home screen.

1.2 Displaying matrices

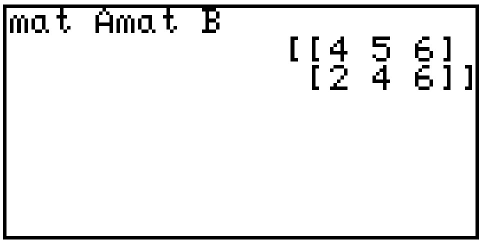
You call up matrices by pressing [MATRIX] [A] (NAME) and the appropriate number. For example, pressing [MATRIX] ([A]) [1] will display 'mat A' on the screen. Pressing [ENTER] will show you the numbers in mat A.



1.3 Multiplying matrices

Press [MATRIX] [1] [MATRIX] [2] to display 'mat A mat B' and press [ENTER] to display the numerical result

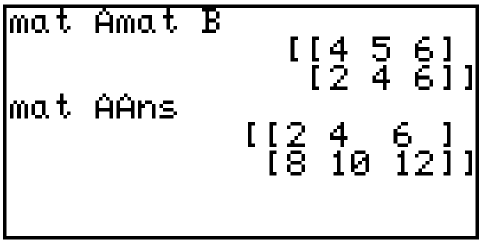
$$\boldsymbol{AB} = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}.$$



The result is stored in *Ans* (as is the result of any calculation).

If you now want to (pre)multiply by $\boldsymbol{A}$ again, i.e. to find $\boldsymbol{A}(\boldsymbol{AB})$, press [MATRIX] [1] [2ndF] [Ans] (on the [ENTER] key) to display 'mat A Ans' and then [ENTER] to give the numerical result

$$\boldsymbol{A}(\boldsymbol{AB}) = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$



Note: If you want to evaluate the expression $\boldsymbol{A}(\boldsymbol{AB})$ using brackets, you need to put in the multiplication sign before the brackets, that is evaluate $\boldsymbol{A}{*}(\boldsymbol{AB})$. This is because a bracket following a matrix, $\boldsymbol{A}(1,2)$ for example, is used to denote the element of a matrix, here the $1,2$ element, i.e. the number in the first row and second column.

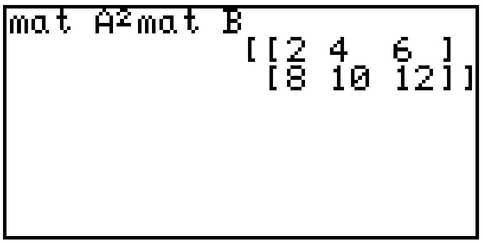
To evaluate $\boldsymbol{A}(\boldsymbol{AB})$, it is much simpler to omit the brackets and just evaluate $\boldsymbol{AAB}$ or $\boldsymbol{A^2B}$.

1.4 Squaring matrices

We could also work out $\boldsymbol{A}(\boldsymbol{AB})$ as $\boldsymbol{A^2B}$.

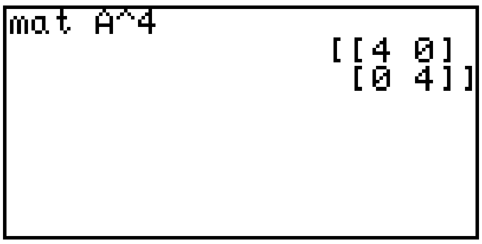
Press [MATRIX] [1] [x^2] [MATRIX] [2] to display 'mat A^2 mat B' and [ENTER] to give the same result

$$\boldsymbol{A}^2\boldsymbol{B} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$



Integer powers of a matrix are produced the same way as with numbers (for positive integers). Only square matrices can be raised to a power.

$$\boldsymbol{A}^4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}.$$



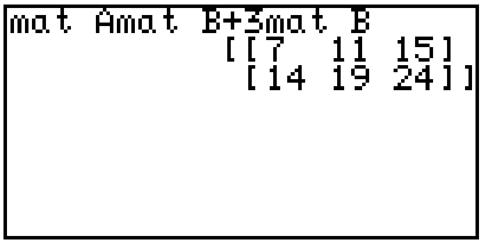
The only negative integer 'power' that works is -1 (using the [x^{-1}] key). However, mat A^{-1} produces the **inverse** of (square) matrix $\boldsymbol{A}$ (see Section 3). The reciprocal of a matrix is not defined.

1.5 More involved expressions

Extensions work just as you would expect.

For example, to work out $\boldsymbol{AB}+3\boldsymbol{B}$, press [MATRIX] [1] [MATRIX] [2] + 3 [MATRIX] [2] to display 'mat A mat B + 3 mat B', and [ENTER] to give

$$\boldsymbol{AB}+3\boldsymbol{B} = \begin{bmatrix} 7 & 11 & 15 \\ 14 & 19 & 24 \end{bmatrix}.$$



Note: If you want to evaluate an expression involving brackets, such as $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})$, you need to put in the multiplication sign before the brackets, that is evaluate $\boldsymbol{A}*(\boldsymbol{B}+\boldsymbol{C})$.

1.6 Storing matrices

If you wanted to keep the previous answer for later use, you might store it in mat C by pressing [STO] [MATRIX] ([A]) [3] [ENTER].

Notice that the calculator automatically makes mat C have the correct order.

1.7 Illegal operations

The calculator will not let you do invalid operations. For example, if you try to calculate $\boldsymbol{A}+\boldsymbol{B}$ by entering [MATRIX] [1] + [MATRIX] [2], you will see 'mat A + mat B' on the screen, but pressing [ENTER] produces the message ERROR 5 Dimension. *Why?*

1.8 Other matrix operations

In the MATRIX D (MATH) menu:

determinant: det mat A = −2 *transpose*: trans mat B = $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

In the MATRIX C (OPE) menu:

- *dimension*: dim mat B = {2 3}
- *'filling' a matrix*: Fill(1, mat A) produces a matrix of '1's: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
- *identity matrix*: identity n produces the $n \times n$ identity matrix. For example,

  identity 2 produces $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

See the calculator guidebook for details on the other MATRIX menu items.

2 Gauss Elimination

2.1 The method

The endpoint of Gauss elimination or Gauss reduction is a matrix in **row-echelon** form, characterised by:

(a) the first non-zero element in each row is a 1 (called a **pivot**);

(b) all elements in the column below a pivot are 0.

To reduce a matrix to row-echelon form, we use **elementary row operations**.[1] The three elementary row operations are

A. exchange two rows ($R_i \rightleftharpoons R_j$)

B. multiply a row by a constant ($R_i \to cR_i,\ c \neq 0$)

C. add a constant multiple of one row to another ($R_i \to R_i + cR_j,\ c \neq 0$).

Procedure

1. If the first column of the matrix is all zeros, 'cross' out this column to leave a smaller matrix.
2. If the element in the first row and first column (top left element) of the matrix is 0, exchange Row 1 with another row (Operation A).

   (Sometimes it is convenient to do this even if the top left element is non-zero, so as to avoid fractions in Step 3.)
3. Multiply the top row by a constant to make the first element 1 (Operation B).
4. Use Operation C to obtain '0's **below** the 1 (pivot) by adding multiples of **Row 1** to each successive row ($R_i \to R_i + cR_1,\ i = 2, 3, \dots$).
5. 'Cross out' the first row and first column to leave a smaller matrix.
6. Go back and start at Step 1 on this smaller matrix.

[1] Each row operation corresponds to the multiplication of the matrix by a corresponding elementary matrix.

Notes

- The only choice in this version of Gauss elimination is which rows to interchange in Step 2. All the other operations are prescribed.
- The solutions to the simultaneous equations described by the row-echelon matrix are the same as those of the original matrix, hence the usefulness of the method. Write down the equations corresponding to the row-echelon matrix and use back substitution to find the solutions.
- The matrix changes wherever we perform an elementary row operation, so that we cannot use equal signs between the steps. Use a $\sim$ instead. The matrices are said to be (row-)equivalent.
- We can proceed in a similar manner to obtain '0' above the pivots too, producing the **reduced row-echelon** matrix. This is called *Gauss-Jordan elimination*. The solution can be read directly from the final matrix, without the need for back substitution.

  The extra effort to do this by hand is not usually worth it, but it is the preferred method if a machine is doing the calculations.

2.2 Using the calculator

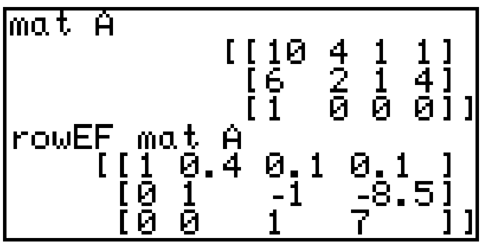
The Sharp 9650/9900 has two built-in commands *rowEF* and *rrowEF* in the MATRIX D (MATH) menu.

rowEF *matrix*, where *matrix* is one of the ten matrices used by the calculator, produces the row-echelon form of *matrix* (Gauss elimination).

rrowEF *matrix* produces the reduced row-echelon form of *matrix* (Gauss-Jordan elimination).

Examples: Gauss elimination

$$\begin{bmatrix} 10 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.4 & 0.1 & 0.1 \\ 0 & 1 & -1 & -8.5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 6 & 2 & 1 & 4 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0.\dot{3} & 0.1\dot{6} & 0.\dot{6} \\ 0 & 1 & 0.25 & 0.25 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

```
mat A
                    [[0 4 1 1]
                     [6 2 1 4]
                     [1 0 0 0]]
rowEF mat A
[[1 0.333333333 0.166…
 [0 1           0.25 …
 [0 0           1    …
```

$$\begin{bmatrix} 0 & 4 & 1 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0.25 & 0.25 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
mat A
                    [[0 4 1 1]
                     [0 2 1 4]
                     [0 1 0 0]]
rowEF mat A
         [[0 1 0.25 0.25]
          [0 0 1    7   ]
          [0 0 0    1   ]]
```

Example: Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 5 & 1 & 3 & 2 \\ 4 & 1 & 1 & 1 \\ 2 & 4 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
rrowEF mat A
                    [[1 0 0 0]
                     [0 1 0 0]
                     [0 0 1 0]
                     [0 0 0 1]]
```

3 Solving Systems of Linear Equations

1. Enter the 4×5 matrix below carefully into mat A using MATRIX EDIT.

$$\begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix}$$

2. Matrix mat A is the augmented matrix $[\boldsymbol{P}|\boldsymbol{Q}]$ for the following system of equations: in matrix form $\boldsymbol{P}\boldsymbol{X}=\boldsymbol{Q}$.

$$\begin{aligned} x_2 + x_3 - 2x_4 &= -3 \\ x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 + x_3 - 3x_4 &= -2 \\ x_1 - 4x_2 - 7x_3 - x_4 &= -19 \end{aligned}$$

Solve this system

(a) using the calculator to find the row-echelon form (Gaussian reduction), and then back substitution (you have to do this).

(b) using the calculator to find the reduced row-echelon form (Gauss-Jordan reduction), from which you can read off the answer.

(c) by using $\boldsymbol{X}=\boldsymbol{P}^{-1}\boldsymbol{Q}$. *Do you understand why this works?*
A little manipulation on the calculator allows us to evaluate $\boldsymbol{P}^{-1}\boldsymbol{Q}$. Change mat A to the matrix $\boldsymbol{P}$ by changing the column dimension to 4 using MATRIX EDIT: this chops off the last column of mat A. Store the column matrix $\boldsymbol{Q}$ in mat B and then evaluate mat A^{-1}mat B.

3. Solve the systems of equations below using each of the three methods. Check your answers by substituting them back into the equations. Can you use Method 2(c) to solve these? What happens in (b) and (c) below? Can you explain? What is $\det(\boldsymbol{P})$? What does this tell you about the inverse matrix?

(a)

$$\begin{aligned} 2x - 5y + 5z &= 17 \\ x - 2y + 3z &= 9 \\ -x + 3y &= -4 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$$

(c)

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ x_1 + x_3 &= 6 \\ 2x_1 - 3x_2 + 5x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

4 Eigenvalues and Eigenvectors

Given an $n \times n$ matrix $\boldsymbol{A}$, the eigenvalues of $\boldsymbol{A}$ are the constants λ and the eigenvectors of $\boldsymbol{A}$ are the corresponding n-dimensional vectors $\boldsymbol{X}$ ($n \times 1$ matrices) satisfying the equation

$$\boldsymbol{A}\boldsymbol{X} = \lambda \boldsymbol{X}.$$

The eigenvalues λ are the zeros or roots of the nth-degree characteristic polynomial

$$p(x) = \det\big(\boldsymbol{A} - x\boldsymbol{I}\big),$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix.

A nice way to find the eigenvalues is graphically: use the EIGENVAL program, which graphs the characteristic polynomial $p(x)$ of the 3×3 matrix stored in mat A. Use *X_Incpt* (in the CALC menu) repeatedly to find the zeroes of this polynomial: these are the eigenvalues λ_i, $i = 1, 2, \ldots$.

To find the eigenvectors, we have to solve the homogeneous matrix equation

$$\big(\boldsymbol{A} - \lambda_i \boldsymbol{I}\big)\boldsymbol{X} = \boldsymbol{0}$$

for each eigenvalue λ_i found above. We do this using Gauss or Gauss-Jordan elimination, most simply done using the commands *rowEF* or *rrowEF* on the matrix $\boldsymbol{A} - \lambda_i \boldsymbol{I}$ for each eigenvalue λ_i.

Example

Find the eigenvalues and corresponding eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}.$$

This is a simple problem that can be done easily by hand. We use it to illustrate the calculator method, which can be used for more complicated problems.

The matrix $\boldsymbol{A}$ is 2×2 matrix, so we need to turn it into a 3×3 matrix for the program: this we do by adding a row and column of zeroes.

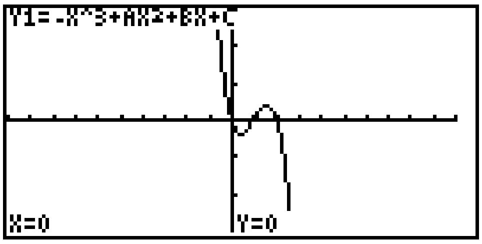
$$\text{mat A} = \begin{bmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This will introduce an extraneous eigenvalue 0, but the other two eigenvalues of mat A will be the eigenvalues of $\boldsymbol{A}$.

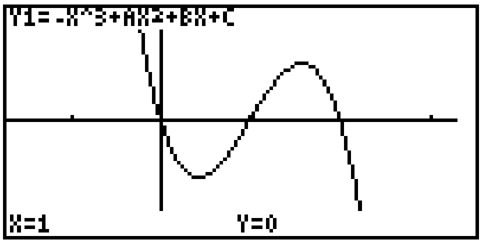
Eigenvalues

Put mat A in your calculator and run EIGENVAL: press [PRGM] [A] (EXE) and the number opposite EIGENVAL.

The program plots the characteristic polynomial $p(x)$, here a cubic, and it is clear that the zeroes lie in the range $0 \leq x < 3$.



Press [ENTER] to end the program, [GRAPH] to replot the graph and ZOOM In on this region so that you can see the graph cutting the x axis. Using *X_Incpt* gives the eigenvalues as $\lambda_1 = 1$ and $\lambda_2 = 2$, together with the extraneous eigenvalue 0 introduced by our method.



In this simple case, it is easy to show by hand that the characteristic polynomial is $p(\lambda) = \lambda^2 - 3\lambda + 2$, with zeroes 1 and 2.

Eigenvectors

$\boldsymbol{\lambda_1 = 1}$: We have to find $\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, such that $\boldsymbol{AX}=\boldsymbol{X}$, or equivalently $\big(\boldsymbol{A}-\boldsymbol{I}\big)\boldsymbol{X} = \boldsymbol{0}$.

Change mat A back to a 2×2 matrix by changing its dimensions using [MATRIX] EDIT. Press the Home key and type

rowEF(mat A−identity 2)

to give the row-echelon form of $\boldsymbol{A}-\boldsymbol{I}$ as

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Note that the bottom row is all zeroes. There must be at least one row of zeroes (the bottom row) in all eigenvector problems.

The corresponding matrix equation is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The bottom row tells us that x_2 is arbitrary, so we set $x_2 = t$, where t is any number. The top row tells us that $x_1 + x_2 = 0$, so that $x_1 = -x_2 = -t$. The eigenvector $\underset{\sim}{\boldsymbol{v}}_1$ corresponding to $\lambda_1 = 1$ is therefore

$$\underset{\sim}{\boldsymbol{v}}_1 = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The eigenvectors are arbitrary (non-zero[2]) multiples of the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We usually say that this vector is the eigenvector, with the understanding that all non-zero multiples of it are also eigenvectors.

Check:

$$\boldsymbol{AX} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \boldsymbol{X}.$$

$\boldsymbol{\lambda_2 = 2}$: We have to solve $\boldsymbol{AX} = 2\boldsymbol{X}$ or $\big(\boldsymbol{A}-2\boldsymbol{I}\big)\boldsymbol{X} = \boldsymbol{0}$. Therefore, execute rowEF$\big($mat A$-$2 identity 2$\big)$ to give the row-equivalent row-echelon form of $\boldsymbol{A}-2\boldsymbol{I}$ as

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $x_2 = t$, where t is any number, and $x_1 + 2x_2 = 0$, so that $x_1 = -2x_2 = -2t$. The eigenvector $\underset{\sim}{\boldsymbol{v}}_2$ corresponding to $\lambda_2 = 2$ is therefore

$$\underset{\sim}{\boldsymbol{v}}_2 = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The eigenvector is an arbitrary (non-zero) multiple of the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Check:

$$\boldsymbol{AX} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2\boldsymbol{X}.$$

[2]If $t = 0$, we obtain a zero vector, which, by definition, cannot be an eigenvector.

Exercise: Find the eigenvalues and corresponding eigenvectors of

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

5 Answers

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2. $x_1 = -1$, $x_2 = 2$, $x_3 = 1$, $x_4 = 3$.

3. **(a)** $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

(b) $x_1 = 1 + 2t$, $x_2 = 2 + 3t$, $x_3 = t$; t any non-zero number.

(c) No solution.

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The eigenvalues are -1, 1 and 2, with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$