# Introduction to Complex Numbers

**Colin Pask**



# Contents

## Contact for more help

Peter McIntyre

School of Mathematics and Statistics
University College (UNSW)
Australian Defence Force Academy
Canberra ACT 2600

Email: p.mcintyre@adfa.edu.au

Phone: (02) 6268 8896

FAX: (02) 6268 8886

# 1 Introduction

We assume that the properties of the set $\mathbb{R}$ of real numbers are understood. In passing, we recall that

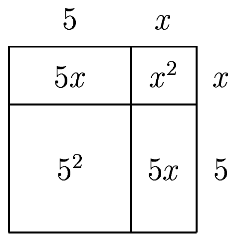
$$\begin{array}{rl} \text{the integers} & 1, 2, 3, \ldots \\ \text{the rationals} & \frac{1}{2}, \frac{3}{7}, \ldots \\ \text{the irrationals} & \sqrt{2}, \pi, \ldots \\ \text{zero} & 0 \\ \text{and negatives} & -1, -\sqrt{2}, \ldots \end{array}$$

were gradually introduced over the centuries and give us a continuum of numbers that can be represented by points on an infinite line:

We also recall that ancient mathematicians used geometric methods or thinking to do what today we call algebra.

**Example:** $x^2 + 10x = 39$ viewed as areas and "completing the square" shows us that

$$\begin{array}{rrcl} & x^2 + 5x + 5x + 25 & = & (5+x)^2, \\ \text{so that} & 39 + 25 & = & (5+x)^2 \\ & 64 & = & (5+x)^2, \end{array}$$




giving $8 = 5 + x$ and so $x = 3$.
Notice the negative solution $x = -13$ is missed here.

However, the quadratic equation, although solved quite generally in ancient times, presented some extreme problems.

**Example:** Solve $z^2 - 2z + 2 = 0$.

The quadratic formula gives $z = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 2}}{2}$

$$\text{or } z = \frac{2 \pm \sqrt{-4}}{2}.$$

What to do about $\sqrt{-4}$?

*One answer is to say that the equation has no solution — and that is quite correct using the concept of number assumed so far.*

However, in the 16th century, mathematicians in Italy began to make some strange discoveries. For example, around 1545 we know that the famous Italian mathematician Cardano explored the problem: "Divide 10 into 2 parts whose product is 40".

This is formulated as

$$\begin{aligned} \text{so} \quad x(10-x) &= 40. \\ \text{Cardano obtained the solution} \quad x &= 5-\sqrt{-15} \\ 10-x &= 5+\sqrt{-15}. \end{aligned}$$

Although bewildered by these answers, he did a check and "putting aside the mental tortures involved" he found:

$$\begin{aligned} (5+\sqrt{-15})(5-\sqrt{-15}) &= 5^2-(\sqrt{-15})^2 \\ &= 5^2 - -15 \\ &= 40. \end{aligned}$$

Thus the answer seemed to have some mathematical sense, even if it was not physically sensible. Cardano concluded: "so progresses arithmetic subtlety the end of which, as is said, is as refined as it is useless."

And there the matter might have ended. But 500 years later we now know Cardano's conclusion about this "useless" mathematics is dramatically wrong. Even in his time the evidence began to appear, showing that these strange "numbers" could actually lead to useful final results.

The first major advance in mathematics beyond that known in ancient times is often said to be the general solution of the cubic equation

$$x^3 = px + q,$$

where $p$ and $q$ are any real numbers; $p,\ q \in \mathbb{R}$.

(You can easily check that a variable change and a little manipulation converts any cubic to that form. As an example, you may show that solving $y^3+3y^2+2y+5=0$ for $y$ is equivalent to solving $x^3 = x-5$ for $x$ by putting $y = x-1$.)

Cardano himself published a formula for the solution to such equations (more on the history of that later).

One of the solutions is given by the formula as

$$x = \sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}.$$

Now those early Italian mathematicians had found that the example

$$x^3 = 15x + 4$$

has solutions $x = -2+\sqrt{3},\ -2-\sqrt{3}$ and 4.

The above formula was supposed to give the answer 4, but instead it produces

$$\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}.$$

In 1560, Bombelli played with the square roots as Cardano had done and reduced the formula's answer to

$$(2+\sqrt{-1})+(2-\sqrt{-1})=4.$$

*Similar manipulations of strange things to end up with correct "real answers" has become a standard and major tool in modern mathematics as we shall see in later examples.*

We still have Bombelli's comment:

> It was a wild thought, in the judgement of many; and I too was for a long time of the same opinion. The whole matter seemed to rest of sophistry rather than on truth. Yet I sought so long until I actually proved this to be the case.

We shall now see how to systematically deal with these problems and then develop the algebra and applications of the new numbers involved. Then we return to polynomial equations, their properties and solutions.

## 2 The Big Invention

All our difficulties really reduce to dealing with $\sqrt{-1}$.

For example, $\sqrt{-4} = \sqrt{4 \times (-1)} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1}$

$$\sqrt{-121} = \sqrt{121 \times (-1)} = \sqrt{121}\sqrt{-1} = 11\sqrt{-1}$$

| | | | | |
|---|---|---|---|---|
| We *define*: | the imaginary unit | $i$ | $=$ | $\sqrt{-1}$ |
| | with the special property | $i^2$ | $=$ | $-1$. |

Thus $\sqrt{-4} = 2i$.

Since $\sqrt{4} = +2$ or $-2$, there is also then the second solution

$$\sqrt{-4} = \sqrt{4 \times (-1)} = -2i.$$

Now the equation $z^2 - 2z + 2 = 0$ has solutions $z = \dfrac{2 \pm 2i}{2} = 1 \pm i$.

We have invented a new class of numbers called **COMPLEX NUMBERS**.

The general complex number is usually denoted by the letter $z$.
We use $z \in \mathbb{C}$ to mean $z$ is in the set of complex numbers.

We write $z = a + bi$ where $a \in \mathbb{R},\ b \in \mathbb{R}$.
Thus $z$ is specified by an ordered pair $(a, b)$ of real numbers.

$a$ is called the real part of $z$, written $a = \mathrm{Re}(z)$.

$b$ is called the imaginary part of $z$, written $b = \mathrm{Im}(z)$.

**Examples**

$8 - 5i$ is a complex number with real part 8 and imaginary part $-5$.

61 is a real number.

$18i$ is a "pure imaginary" number.

**Notes**

1. Real numbers are a special set of complex numbers. They are complex numbers with zero imaginary parts.

2 We can now solve all quadratic equations, if solutions in $\mathbb{C}$ are accepted.

### 2.1 Invention or discovery?

We have talked about **inventing** complex numbers, but some people would say that they were there all the time, just waiting to be **discovered** as soon as we explored quadratic equations. There is a long-standing philosophical debate about whether we invent mathematics or whether there is an already pre-existing world of mathematics that we gradually discover. People taking the second view are often called **Platonists** after the ancient Greek philosopher Plato (about 427 BC to 347 BC).

You may think it is all easier if we think about real numbers. But even just the integers can cause us some worries. For example, we might feel we know what 3 apples or 3 cars or 3 pigs means. But exactly what is "3"? How can we define "three-ness"? And we might feel comfortable with 3 pigs minus 2 pigs means 1 pig, but how about 3 pigs minus 5 pigs?

We better move on before philosophical doubts overwhelm us!

## 3 Manipulating Complex Numbers

This will be just like doing ordinary algebra, as for example in

$$\begin{aligned} (1+x)(3+x) &= 3 + x + 3x + x^2 && \text{multiply out} \\ &= 3 + 4x + x^2 && \text{gather up,} \end{aligned}$$

except now $i$ is involved and

**every time we get an $i^2$, we replace it with $-1$.**

O.K., let's go through the usual operations. $a$, $b$, $c$ and $d$ stand for our usual "real numbers" and always obey our rules for real numbers.

1. **Equality:** $a + bi = c + di$ requires $a = c$ and $b = d$,
   i.e. equal real parts **and** equal imaginary parts.

2. **Addition:** $(a + bi) + (c + di) = (a + c) + (b + d)i$.

3. **Subtraction:** $(a + bi) - (c + di) = (a - c) + (b - d)i$.

   Notice that we have been "gathering up the real parts and the imaginary parts."

   **Example:** Does $(3 + 4i) + (2 + 3i) - 2i$ equal $5 + 9i$?

   Left-hand side is $(3 + 2) + (4 + 3 - 2)i = 5 + 5i$.

   *Answer*: No. The real parts are equal, but the imaginary parts are not.

   The statement would be correct if we had $+2i$ instead of $-2i$.

**4. Multiplication:**
$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2\\ &= ac+(ad+bc)i-bd\\ &= (ac-bd)+(ad+bc)i.\end{aligned}$$

Notice that in all these cases, combining two complex numbers gives another complex number, and to find the real and imaginary parts we are just using the usual rules for real or ordinary numbers, plus the fact that $i^2=-1$.

*An aside*: the formula for multiplication is messy and strange. It would have been much nicer if $(a+bi)(c+di)$ somehow gave $ac+bdi$ but it does not. Later we will find a much nicer and "more sensible" way to view multiplication.

**Examples**

$$\begin{aligned}3(4+2i) &= 3\times 4+3\times 2i = 12+6i.\\ (2+i)(4+3i) &= 2\times 4+2\times 3i+4i+3i^2\\ &= 8+6i+4i-3\\ &= 5+10i.\\ 2i(4+3i) &= 8i+6i^2\\ &= -6+8i.\end{aligned}$$

The powers of $i$ are interesting:

| | | | |
|---|---|---|---|
| $i$ | $i^2=-1$ | $i^3=(i^2)i=-i$ | $i^4=(i^2)^2=(-1)^2=1$ |
| $i^5=i$ | $i^6=-1$ | $i^7=-i$ | $i^8=1$ |

and the pattern $i \quad -1 \quad -i \quad 1$ repeats.

*The next obvious operation to discuss is division, but, before doing that, it is useful to introduce a new operation that has no counterpart in real-number theory.*

**5. Complex conjugation**

If $z$ is a complex number, its **complex conjugate** is denoted by $\overline{z}$.

> If $z=a+bi$, its complex conjugate is $\overline{z}=a-bi$.
>
> To find the complex conjugate of $z$, just change the sign of its imaginary part.

**Examples:** $\overline{3+4i}=3-4i \qquad \overline{8}=8 \qquad \overline{-5i}=5i \qquad 2+i=\overline{2-i}.$

This is a very simple operation, but it turns out to be extremely useful and important, especially as the study of complex numbers extends into "complex analysis" and is widely applied in science and engineering.

It is easy to establish a few simple properties and results.

1. If $z$ is a real number, $z = \overline{z}$.
2. If $z$ is pure imaginary, $z = -\overline{z}$.
3. If $z = a + bi$, so that $\overline{z} = a - bi$,
   $z + \overline{z} = 2a$ and $z - \overline{z} = 2bi$.
   So $\boxed{\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z}) \qquad \mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})}$.
4. $z\overline{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$ (as $i^2 = -1$).
   i.e. $\boxed{z\overline{z} = a^2 + b^2, \text{ a real quantity}}$

**6. Division**

As usual, $z = a + bi, \quad a, b \in \mathbb{R}$.

If $c$ is a real number, division is as usual — multiplying by $1/c$, which we know how to do:

$$\frac{z}{c} = \left(\frac{1}{c}\right) z = \frac{1}{c}(a + bi) = \left(\frac{a}{c}\right) + \left(\frac{b}{c}\right) i.$$

Thus dividing by a real number means dividing both the real and imaginary parts by that number.

Next, let $\quad w = c + di \quad c, d \in \mathbb{R}$.

We now need $\dfrac{z}{w} = \dfrac{a + bi}{c + di}$.

We use a trick to convert this to a multiplication of complex numbers and a division by a real number, both of which operations we can do.

*The problem is the di part on the bottom. Remember you had a similar problem in maths at school when dividing by $c + \sqrt{d}$. You got rid of square roots in the denominator by "multiplying top and bottom by $c - \sqrt{d}$"*, called rationalising the denominator.

We know that multiplying $w$ by its complex conjugate $\overline{w}$ will give us a real number, so we form

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{(a + bi)(c - di)}{c^2 + d^2}$$

multiplication of two complex numbers
and division by the real number $c^2 + d^2$.

**Example**

$$\begin{aligned}\frac{2+i}{3+2i} &= \frac{(2+i)(3-2i)}{(3+2i)(3-2i)} \\ &= \frac{6-4i+3i-2i^2}{3^2+2^2} \\ &= \frac{6+(-4+3)i--2}{13} \\ &= \frac{8-i}{13} \\ &= \left(\frac{8}{13}\right)-\left(\frac{1}{13}\right)i.\end{aligned}$$

**Example:** Dividing by $i$

$$\begin{aligned}\frac{6+5i}{i} &= \frac{(6+5i)(-i)}{(i)(-i)} \\ &= \frac{-6i-5i^2}{-i^2} \\ &= \frac{-6i--5}{-(-1)} \\ &= 5-6i.\end{aligned}$$

**Example:** $\dfrac{10}{2+i}=\dfrac{10(2-i)}{2^2+1^2}=4-2i.$

**Observations**

1. Putting $z=1$ gives the inverse of $w$:

$$w^{-1}=\frac{1}{w}=\frac{\overline{w}}{(w\overline{w})} \quad \text{or} \quad (c+di)^{-1}=\frac{c-di}{c^2+d^2}.$$

   We can then check: $ww^{-1}=\dfrac{w\overline{w}}{w\overline{w}}=1.$

2. If $c^2+d^2=0$, both $c$ and $d$ must be zero.

   If $w=0+0i$, we cannot divide by $w$.

   The complex number zero is $0+0i$.

*This is all straightforward algebra once you let your imagination loose, accept the definition of* $i = \sqrt{-1}$ *and replace* $i^2$ *by* $-1$ *where necessary.*

The formal properties or laws that we can write down for real numbers also hold for complex numbers, as long as we remember that zero is now $0 + 0i$.

*We now ask another question which will develop our mathematics a little more and give us a new approach to complex numbers. Using the algebra-geometry link, we will be led to new representations of complex numbers, and in one of them the operations of multiplication and division will be neater and make more sense.*

## 4 How Big is a Complex Number?

To answer this question, we need a new way of representing complex numbers.

We "look at them" by extending the **number line** to the **complex plane**.

If $z = a + ib$, it can be **represented** by a point $(a, b)$ in the **complex plane**

So the complex plane is like our usual Cartesian or $xy$ plane, except now the $x$ axis is the axis used for the real part of $z$ and the $y$ axis is the axis used for the imaginary part of $z$.

Here is an example of some numbers plotted in the complex plane.

This diagram will help us answer the question: how big is a complex number?

Somehow 5 and $-5$ seem equally big — in terms of distance travelled from zero along the number line.

Similarly $5i$ and $-5i$ seem equally big — the distance travelled from zero or the origin is the same, but now we go along the imaginary axis.

But all the other complex numbers plotted are the same distance from the origin: in each case Pythagoras' Theorem and $\sqrt{3^2+4^2}=5$ or $\sqrt{4^2+3^2}=5$ confirm it.

This suggests that all those numbers are equally big, as they are the same distance from the origin.

Generally, we see that the complex number $z=a+bi$, when plotted in the complex plane, gives a point that is distance $\sqrt{a^2+b^2}$ from the origin.

We now define:

> The size or **modulus** of a complex number $z$ is denoted by $|z|$.
> If $z=a+bi$, $|z|=\sqrt{a^2+b^2}$.

All the complex numbers with $|z|=5$ lie on a circle of radius 5; some of them are plotted above.

We now see how **complex conjugate, modulus, inverses** and **division** are related:

$$\begin{array}{llcl}
\text{We saw that} & z\overline{z} = a^2 + b^2 & \text{so} & |z|^2 = z\overline{z} \\
 & w^{-1} = \dfrac{1}{w} = \dfrac{\overline{w}}{w\overline{w}} & \text{so} & w^{-1} = \dfrac{\overline{w}}{|w|^2} \\
 & \dfrac{z}{w} = \dfrac{z\overline{w}}{w\overline{w}} & \text{so} & \dfrac{z}{w} = \dfrac{z\overline{w}}{|w|^2}
\end{array}$$

## 4.1 A new algebraic representation for complex numbers

A point in a plane can be specified using the usual rectangular coordinates or by giving distance $r$ from the origin and going anticlockwise round an angle $\theta$ from the $x$ or real axis. These are usually called polar coordinates.

This gives us the following

Rectangular Form: $z = a + ib$

$a = r\cos\theta$
$b = r\sin\theta$

$r =$ modulus $= |z| = \sqrt{a^2 + b^2}$
$\theta =$ argument of $z$, written $\arg(z)$
$\tan\theta = b/a$
Principal argument $\text{Arg}(z)$ has $-\pi < \theta \leq \pi$

Polar form: $z = r(\cos\theta + i\sin\theta)$
$z = r\,\text{cis}(\theta)$

*Notice the shorthand notation* $\mathbf{cis}(\boldsymbol{\theta}) = \mathbf{cos}\,\boldsymbol{\theta} + \boldsymbol{i}\,\mathbf{sin}\,\boldsymbol{\theta}$.

This gives us a new way of **representing** a complex number in terms of a length $r$ and an angle $\theta$.

*This representation and the underlying geometrical idea or representation turn out to be extremely useful.*

Here are some examples showing how the different representations work. First the geometry, then the algebra.

The real number $z_1 = 2$ is on the real axis, so $\theta = 0$ and

$$z_1 = 2(\cos 0 + i \sin 0) = 2 \operatorname{cis}(0).$$

The pure imaginary number $z_2 = 2i$ is on the imaginary axis, so $\theta = \pi/2$ and

$$z_2 = 2i = 2 \operatorname{cis}(\pi/2).$$

The complex number $z_3$ in rectangular form is

$$z_3 = 2 + 2i.$$

To put it into the polar representation $r\operatorname{cis}(\theta)$, we need to find the modulus $|z_3| = r$ and the angle or argument $\arg(z_3) = \theta$.

$$|z_3| = \sqrt{2^2 + 2^2} = 2\sqrt{2}.$$

We can now write $z_3 = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$ and, recognising $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, we find

$$z_3 = 2\sqrt{2}\big(\cos(\pi/4) + i \sin(\pi/4)\big) = 2\sqrt{2} \operatorname{cis}(\pi/4).$$

*Notice that we could have used* $\tan(\theta) = 2/2 = 1$ *to get* $\theta = \pi/4$. *We need to check on the diagram that* $\tan\theta = 1$ *does mean* $\theta = \pi/4$ *here, and not* $\theta = -3\pi/4$.

Similarly, we can find

$$\begin{aligned}
\overline{z}_3 &= 2-2i &= 2\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right) &= 2\sqrt{2}\text{cis}(-\pi/4) \\
z_4 &= -2+2i &= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) &= 2\sqrt{2}\text{cis}(3\pi/4) \\
\overline{z}_4 &= -2-2i &= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right) &= 2\sqrt{2}\text{cis}(-3\pi/4).
\end{aligned}$$

**Important Observations**

1. Plotting complex numbers in the complex plane is a great help in finding the angles involved.
2. We have chosen the angle $\theta$ to be in the range $-\pi < \theta \leq \pi$. That is a convention, although other angles (e.g. $5\pi/4$ instead of $-3\pi/4$) are equally valid.
3. Changing the sign of the imaginary part is equivalent to a reflection in the real axis or changing the sign of the angle. This means

   $$\boxed{\text{if } z = r\,\text{cis}(\theta) \quad \text{complex conjugate } \overline{z} = r\,\text{cis}(-\theta)}\,.$$

   Changing the sign of the real part is equivalent to a reflection in the imaginary axis, e.g. $z_3$ and $z_4$ in the previous figure.

   Changing the sign of the whole complex number is equivalent to a reflection through the origin, e.g. see $z_3$ and $\overline{z_4}$ in the previous figure, where $\overline{z_4} = -z_3$.
4. Recall that the number line helps us to solve problems like

   Find all the numbers $x$ with $|x-1| < 2$ and $|x+1| < 1$.

   Areas in the complex plane can be used in a similar way.

   **Example:** The complex numbers $z$ with $|z| < 2$ all lie inside a circle radius 2, centred on the origin. Those with $\text{Re}(z) > 1$ all lie in the complex plane to the right of the vertical line through 1 on the real axis.

**Examples**

The line shows complex numbers $z$ with $\text{Re}(z) \leq 2$ and $\text{Re}(z) = \text{Im}(z)$.

The shaded semi-circular area shows all complex numbers with $|z| \leq 2$ and $\text{Im}(z) \geq 0$. (The lines enclosing the shaded area would **not** be included if it was $|z| < 2$ and $\text{Im}(z) > 0$.)

*We now move on to manipulating complex numbers using the polar representation. We will find some crucial advantages for both understanding and developing the theory of complex numbers.*

## 4.2 Equality in different representations

If

$$\begin{aligned} z_1 &= a_1 + b_1 i &= r_1 \operatorname{cis}(\theta_1), \\ z_2 &= a_2 + b_2 i &= r_2 \operatorname{cis}(\theta_2), \end{aligned}$$

$z_1 = z_2$ means

$$\begin{aligned} &a_1 = a_2 \quad \text{and} \quad b_1 = b_2 \\ &r_1 = r_2 \quad \text{and} \quad \theta_1 = \theta_2 \quad \text{or} \quad \theta_1 = \theta_2 + 2m\pi \quad m \text{ an integer.} \end{aligned}$$

Rotating by any multiple of $2\pi$ around the origin in the complex plane gets back to the same point.

If we insist that $-\pi < \theta \leq \pi$, i.e. we use the principal argument $\text{Arg}(z)$, two complex numbers are equal if

they have the same modulus
and they have the same principal argument.

## 4.3 Multiplication

If $z_1 = r_1 \operatorname{cis}(\theta_1)$ and $z_2 = r_2 \operatorname{cis}(\theta_2)$,

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos(\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\big(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\big) \\ &= r_1 r_2\Big(\big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\big) + i\big(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\big)\Big). \end{aligned}$$

$$\boxed{\begin{aligned} z_1 z_2 &= r_1 r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big) \\ &= r_1 r_2 \operatorname{cis}(\theta_1+\theta_2) \end{aligned}}$$

To multiply two complex numbers, we
multiply their sizes or moduli
and add their arguments.

It is satisfying to see that the size of the product is related to the product of the sizes or moduli of the complex numbers being multiplied together. That is not at all obvious in the rectangular representation — see Section 3 (iv).

## 4.4 Complex conjugate

If $z = a + bi = r(\cos\theta + i\sin\theta)$,

$\overline{z} = a - bi = r(\cos\theta - i\sin\theta)$.

Since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin(\theta)$, this can be written as

$$\overline{z} = r\big(\cos(-\theta) + i\sin(-\theta)\big).$$

$$\boxed{z = r\operatorname{cis}(\theta) \qquad \overline{z} = r\operatorname{cis}(-\theta)}$$

Geometrically, taking the complex conjugate is equivalent to reflection in the real axis, as the examples in Section 4.1 illustrate.

## 4.5 Division

Since in this representation, if $z_2 = r_2 \operatorname{cis}(\theta_2)$,

$$\overline{z}_2 = r_2 \operatorname{cis}(-\theta_2),$$

we easily find

$\dfrac{z_1}{z_2} = \dfrac{z_1\overline{z}_2}{z_2\overline{z}_2} = \dfrac{z_1\overline{z}_2}{|z_2|^2}$ gives us

$$\frac{z_1}{z_2} = \frac{r_1 \operatorname{cis}(\theta_1)}{r_2 \operatorname{cis}(\theta_2)} = \frac{r_1 \operatorname{cis}(\theta_1)\, r_2 \operatorname{cis}(-\theta_2)}{r_2^2}.$$

$$\boxed{\therefore\ \frac{z_1}{z_2} = \frac{r_1}{r_2}\Big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\Big) = \frac{r_1}{r_2}\operatorname{cis}\big(\theta_1 - \theta_2\big)}$$

In this case, we divide the sizes or moduli (*again a natural or desirable result*) and subtract the arguments.

A little consistency check:

$$\begin{aligned}
\left(\frac{z_1}{z_2}\right) z_2 &= \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)\ \cdot\ r_2 \operatorname{cis}(\theta_2) && \text{division rule} \\
&= \frac{r_1}{r_2}\, r_2 \operatorname{cis}(\theta_1 - \theta_2 + \theta_2) && \text{multiplication rule} \\
&= r_1 \operatorname{cis}(\theta_1) \\
&= z_1.
\end{aligned}$$

## 4.6 Summary

The new modulus-argument representation suggested by using the complex plane to picture complex numbers has given us neat forms for complex conjugation, multiplication and division, and also satisfying interpretations of the last two operations.

The modulus-argument representation does not give us anything better in terms of addition and subtraction. The geometrical representation does show us the similarity with vector addition and subtraction.

**Example:** For $z_1 + z_2 = (a + bi) + (c + di) = (a + b) + (c + d)i$

*We are now ready to establish a major result in complex analysis.*

# 5 De Moivre's Theorem

The multiplication rule in Section 4.3 gives $z^2 = r\,\text{cis}(\theta)\, r\,\text{cis}(\theta) \;=\; r^2\,\text{cis}(2\theta)$.

Then $z^3 = r\,\text{cis}(\theta)\, r^2\,\text{cis}(2\theta) \;=\; r^3\,\text{cis}(3\theta)$.

Repeating this $n$ times gives De Moivre's Theorem.

$$\boxed{\begin{array}{l}\text{If } z = r(\cos\theta + i\sin\theta) = r\,\text{cis}(\theta),\\ \quad z^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) = r^n\,\text{cis}(n\theta).\end{array}}$$

(A formal proof using Mathematical Induction can be given for this theorem.)

Abraham De Moivre (1667–1754) was a French mathematician who did important work on the theory of probability. Because of religious upheavals (he was a Protestant), he fled to England at age 18.

*This theorem has all sorts of uses and we shall consider three of them.*

## 5.1 Raising complex numbers to an integer power

This is very simple if we use De Moivre's Theorem. The only work may be to convert to polar form.

**Example:** Find $(1+i)^9$.

Since $|1+i| = \sqrt{2}$, we can write $1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$,

so $1+i = \sqrt{2}\big(\cos(\pi/4) + i\sin(\pi/4)\big)$.

We now know immediately from De Moivre's Theorem that

$$\begin{aligned}(1+i)^9 &= (\sqrt{2})^9\Big(\cos(9\pi/4) + i\sin(9\pi/4)\Big) \\ &= 16\sqrt{2}\Big(\cos(\pi/4+2\pi) + i\sin(\pi/4+2\pi)\Big), \\ \text{so} \quad (i+i)^9 &= 16\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right).\end{aligned}$$

$$\boxed{(1+i)^9 = 16+16i}$$

We can **visualise** this process: $(1+i)$ is multiplied by $(1+i)$ eight times. Each multiplication increases the modulus by $\sqrt{2}$ and rotates the angle by $\pi/4$.

## 5.2 Finding the $n$th root of a complex number

We say that $w$ is an $n$th root of $z$ if $w^n = z$.

$$\begin{array}{lrcl} \text{Suppose} & z & = & r(\cos\theta + i\sin\theta) \\ \text{and let} & w & = & s(\cos\beta + i\sin\beta). \end{array}$$

Using De Moivre's Theorem to find $w^n$ then gives

$$s^n\big(\cos(n\beta) + i\sin(n\beta)\big) = r\big(\cos\theta + i\sin\theta\big).$$

Now two equal complex numbers must have equal moduli, so

$$s^n = r \quad \text{or} \quad \boxed{s = \sqrt[n]{r} = r^{1/n}}$$

This determines the modulus of the $n$th root.

The arguments must also be equal — **but only up to a multiple of $2\pi$** (as explained in Section 4.2), i.e.

$$n\beta = \theta + 2\pi k \qquad k = 0, 1, 2, 3, \ldots.$$

$$\boxed{\beta = \frac{\theta}{n} + 2\pi\frac{k}{n}}$$

But what values to take for $k$?

Perhaps the easiest way to answer this is to try out some cases.

| | | | | | | |
|---|---|---|---|---|---|---|
| $k \rightarrow$ | $0$ | $1$ | $2$ | $3$ | $\ldots\ldots\ldots$ | $n-1$ |
| $2\pi\frac{k}{n} \rightarrow$ | $0$ | $\frac{2\pi}{n}$ | $2\pi\frac{2}{n}$ | $2\pi\frac{3}{n}$ | $\ldots\ldots\ldots$ | $2\pi\frac{(n-1)}{n}$ |
| $k \rightarrow$ | $n$ | $n+1$ | $n+2$ | $n+3$ | $\ldots\ldots\ldots$ | $2n-1$ |
| $2\pi\frac{k}{n} \rightarrow$ | $2\pi$ | $2\pi + \frac{2\pi}{n}$ | $2\pi + 2\pi\frac{2}{n}$ | $2\pi + 2\pi\frac{3}{n}$ | $\ldots\ldots\ldots$ | $2\pi + 2\pi\frac{(n-1)}{n}$ |

i.e. just the same angles again with an extra (meaningless) $2\pi$ included.

Similarly, $k = 2n$, $(2n+1)$, $\ldots$ generates the same angles again, but with $4\pi$ added, then $6\pi$ added and so on.

In fact only $k = 0$, $1$, $2$, $\ldots (n-1)$ gives truly different angles. Thus we have found

If $z = r(\cos\theta + i\sin\theta)$, then $z$ has exactly $n$ distinct $n$th roots given by

$$\sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right),$$

with $k = 0, 1, 2, \ldots (n-1)$.

There is a very neat and beautiful picture of the roots of complex numbers when we plot them in the complex plane.

The $n$th roots of the complex number $z = r\,\mathrm{cis}(\theta)$ are equally spaced by angle $2\pi/n$ around a circle of radius $\sqrt[n]{r}$, with the first root having argument $\theta/n$.

## 5.3 Examples

1. **The cube roots of unity**

   $z = 1 = 1\big(\cos(0) + i\sin(0)\big)$, so $r = 1$ and $\theta = 0$.

   Then the roots are $\sqrt[3]{1}\left(\cos\left(\dfrac{0 + 2\pi k}{3}\right) + i\sin\left(\dfrac{0 + 2\pi k}{3}\right)\right)$ for $k = 0, 1, 2$.

   $$\begin{aligned} \text{i.e.}\quad k = 0 &: \quad 1 \\ k = 1 &: \quad \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ k = 2 &: \quad \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{aligned}$$

   Try cubing each of these to check that you do get back 1.

   The cube roots of 27 would have the same arguments or angles, but they would lie on a circle of radius 3.

## 2. Solving an equation

Solve $z^4 + 2\sqrt{3} + 2i = 0$.

This means $z = (-2\sqrt{3} - 2i)^{1/4}$.

$|-2\sqrt{3} - 2i| = \sqrt{4 \times 3 + 4} = 4$.

Then $-2\sqrt{3} - 2i = 4\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$.

$$\therefore \quad z = \left(4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\right)^{1/4},$$

$$\text{so} \quad z = 4^{1/4}\left(\cos\left(\frac{\frac{7\pi}{6} + 2\pi k}{4}\right) + i\sin\left(\frac{\frac{7\pi}{6} + 2\pi k}{4}\right)\right), \qquad k = 0,\ 1,\ 2,\ 3.$$

Thus the required solutions are

$$\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{24}\right),\ \sqrt{2}\operatorname{cis}\left(\frac{19\pi}{24}\right),\ \sqrt{2}\operatorname{cis}\left(\frac{31\pi}{24}\right),\ \sqrt{2}\operatorname{cis}\left(\frac{43\pi}{24}\right).$$

### Observations

1. If we had used the principal argument to write $z = 4\operatorname{cis}(-5\pi/6)$ instead of $4\operatorname{cis}(7\pi/6)$, the same roots would be found, but the last one would come out at $\sqrt{2}\operatorname{cis}(-5\pi/24)$ instead of $\sqrt{2}\operatorname{cis}(43\pi/24)$.

2. Geometrically, these roots are easily found on a circle of radius $\sqrt{2}$.

## 5.4 Proving results about trig functions

If we take $r = 1$, De Moivre's Theorem is simply

$$\big(\cos\theta + i\sin\theta\big)^n = \cos(n\theta) + i\sin(n\theta);$$

this can be used to prove useful identities.

**Example:** Find an expression for $\cos(3\theta)$ in terms of $\cos\theta$.

Take $n = 3$ to get

$$\begin{aligned}\cos(3\theta) + i\sin(3\theta) &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta).\end{aligned}$$

Equating the real parts gives

$$\begin{aligned}\cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta),\end{aligned}$$

so that

$$\boxed{\cos(3\theta) = 4\cos^3\theta - 3\cos\theta}$$

Similarly, equating the imaginary parts will give

$$\boxed{\sin(3\theta) = 3\sin\theta - 4\sin^3\theta}$$

**Exercise:** Use De Moivre's Theorem to find the formulas for $\cos(2\theta)$ and $\sin(2\theta)$.

*Isn't it amazing: thinking about* $\sqrt{-1}$ *and suitable representations has lead to trigonometric itentities!*

# 6 Exploring the Algebra: Operations on Combinations of Complex Numbers

We have introduced two new operations — taking the complex conjugate to get $\overline{z}$ from $z$ and taking the modulus to get $|z|$ from $z$.

We now show how this works when $z$ is a combination, sum or product, of complex numbers.

This little section again illustrates the power of using the most suitable representation of $z$.

## 6.1 Modulus of combinations

Our multiplication result in polar form (Section 4.3) immediately tells us

$$\boxed{|z_1 z_2| = |z_1|\,|z_2|}$$

How about $|z_1 + z_2|$?

Since $(a + bi) + (c + di) = (a + c) + i(b + d)$, we can look on adding complex numbers like adding vectors.

If we now look at the triangle and use the geometrical result "length of a side of a triangle is less than or equal to the sum of the lengths of the other two sides"
we immediately get

$$\boxed{|z_1 + z_2| \leq |z_1| + |z_2|} \text{ — "triangle inequality"}$$

Thus the modulus of the sum of two complex numbers is not necessarily equal to the sum of the moduli of those numbers.

**Exercise:** When **does** the equality hold?

## 6.2 Complex conjugate of combinations

We recall that

| | | | |
|---|---|---|---|
| if | $z = a + bi$ | then | $\bar{z} = a - bi$ |
| if | $z = r\,\mathrm{cis}(\theta)$ | then | $\bar{z} = r\,\mathrm{cis}(-\theta)$ |

Using the first representation we find

$$\begin{aligned} z_1 + z_2 &= (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2)i \\ \bar{z}_1 + \bar{z}_2 &= (a_1 - b_1 i) + (a_2 - b_2 i) = (a_1 + a_2) - (b_1 + b_2)i \end{aligned}$$

and so clearly $\boxed{\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2}$

Using the second representation we find

$$\begin{aligned} z_1 z_2 &= r_1\,\mathrm{cis}(\theta_1)\; r_2\,\mathrm{cis}(\theta_2) = r_1 r_2\,\mathrm{cis}(\theta_1 + \theta_2) \\ \bar{z}_1 \bar{z}_2 &= r_1\,\mathrm{cis}(-\theta_1)\; r_2\,\mathrm{cis}(-\theta_2) = r_1 r_2\,\mathrm{cis}(-\theta_1 - \theta_2) \end{aligned}$$

and so clearly $\boxed{\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2}$

These results naturally extend to any number of complex numbers:

*The complex conjugate of a sum of complex numbers = the sum of the individual complex conjugates.*

*The complex conjugate of a product of complex numbers = the product of the individual complex conjugates.*

**Example**

$$\begin{aligned} \overline{(1+i)(2-i)(1+2i)} &= \overline{(1+i)}\;\overline{(2-i)}\;\overline{(1+2i)} \\ &= (1-i)(2+i)(1-2i) \\ &= (1-7i). \end{aligned}$$

# 7 Euler's Formula

We are now ready to look at one of the most famous of all formulas in mathematics. It was developed by the Swiss mathematician Leonard Euler (1707–1783), who was probably the most prolific of all mathematicians. (He introduced the symbol $i$ for $\sqrt{-1}$.)

Suppose for the real variable $\theta$, we define the function

$$f(\theta) = \cos\theta + i\sin\theta.$$

(We have now introduced the idea of a complex-valued function, by the way.)

Then De Moivre's Theorem tells us that

$$\big(f(\theta)\big)^n = \cos(n\theta) + i\sin(n\theta)$$

$$\text{i.e.} \quad \boxed{\big(f(\theta)\big)^n = f(n\theta)}$$

We can also see that the derivative is given by

$$\begin{aligned} f'(\theta) &= -\sin\theta + i\cos\theta \\ &= i^2\sin\theta + i\cos\theta \\ &= i(\cos\theta + i\sin\theta) \\ \text{i.e.} \quad f'(\theta) &= i\,f(\theta). \end{aligned}$$

$$\text{or} \quad \boxed{f'(\theta) = \text{constant} \times f(\theta)} \quad \text{where the constant is } i.$$

*Have you come across another function with those properties?*

How about $e^{ax}$, where $a$ is a constant?

$$\big(e^{ax}\big)^n = e^{nax} = e^{a(nx)}.$$

$$\frac{d}{dx}e^{ax} = ae^{ax}.$$

This leads us to say

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \text{ — Euler's Formula.}$$

Can we check this another way?

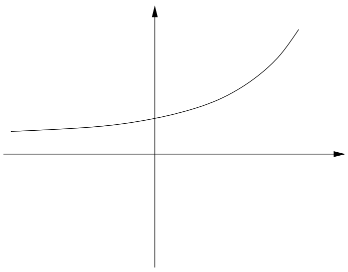
$$\text{We know} \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \quad \text{Taylor series about } x = 0$$

$$\begin{aligned}
\text{so let's try defining} \quad e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \ldots\right) \\
&= \big(\cos\theta\big) + i\big(\sin\theta\big).
\end{aligned}$$

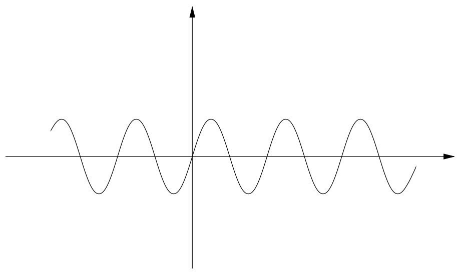
Yes, the series expressions also give

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

*This most remarkable formula links two quite different "worlds": the monotonically changing, unbounded "exponential world" and the bounded, oscillating "sinusoidal world".*



exponential



sine

## 7.1 Some special values

$\theta = 0 \qquad e^{i0} = \cos 0 + i \sin 0 = 1$

$\theta = 2\pi \qquad e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1$

$\theta = k2\pi \qquad e^{ik2\pi} = 1$ also

$\theta = \pi/2 \qquad e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$

$\theta = \pi \qquad e^{i\pi} = \cos\pi + i\sin\pi = -1$

$$\boxed{e^{i\pi} = -1}$$

*This strange formula links some of the most mysterious elements in all of mathematics*

$$e \qquad i \qquad \pi \qquad -1$$

*In a 1990 poll conducted by the magazine The Mathematical Intelligencer, it came out top as the most beautiful result in the whole subject.*

# 8 Using Euler's Formula

This formula $e^{i\theta} = \cos\theta + i\sin\theta$ turns out to have many important applications. They are often used because the rules for manipulating exponentials are so simple:

| | | | |
|---|---|---|---|
| Rule 1 | $e^a e^b = e^{a+b}$ | Rule 2 | $\dfrac{e^a}{e^b} = e^{a-b}$ |
| Rule 3 | $\dfrac{d}{dx}(e^{ax}) = ae^{ax}$ | Rule 4 | $\displaystyle\int e^{ax}\,dx = \frac{1}{a}e^{ax}$ (+ constant) |

## 8.1 Another way to write complex numbers

We can now write $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Then the multiplication and division laws follow simply from Rules 1 and 2 for exponentials:

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)},$$ i.e. *multiply moduli, add arguments*

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)},$$ i.e. *divide moduli, subtract arguments*

De Moivre's Theorem just becomes

$$\boxed{(re^{i\theta})^n = r^n e^{in\theta}}$$

## 8.2 Proving identities

The multiplication rule gives

$$\boxed{e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}}$$

Written out using trigonometric functions it is

$$\begin{aligned}
&\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\\
&= \big(\cos\theta_1 + i\sin\theta_1\big)\big(\cos\theta_2 + i\sin\theta_2\big)\\
&= \cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\\
&= \big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\big) + i\big(\cos\theta_1\cos\theta_2 + \sin\theta_1\cos\theta_2\big)\,.
\end{aligned}$$

Equating real parts and imaginary parts gives the much used identities

$$\boxed{\begin{aligned}
\cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\\
\sin(\theta_1+\theta_2) &= \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2
\end{aligned}}$$

We can now see (in the above boxes) how the exponential and trigonometric worlds are linked.

*Which world do you see as the more simple and elegant?*

## 8.3 Using complex exponentials

It is often very useful to move into the "complex world" to solve a problem and then return to the "real world" with the answer. Actually we already saw an example of that when finding the real roots of cubic equations (Section 1).

Here is one simple way to proceed.

$$\begin{aligned}
\text{Since}\qquad e^{i\theta} &= \cos\theta + i\sin\theta\\
e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta
\end{aligned}$$

we can add and subtract these expressions to write

$$\begin{aligned}
\cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta})\\
\sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).
\end{aligned}$$

**Example**

$$\begin{aligned}
\int \cos^2\theta\, d\theta &= \int \frac{1}{4}\left(e^{i\theta}+e^{-i\theta}\right)^2 d\theta \\
&= \frac{1}{4}\int \left(\left(e^{i\theta}\right)^2+\left(e^{-i\theta}\right)^2+2e^{i\theta}e^{-i\theta}\right) d\theta \\
&= \frac{1}{4}\int \left(e^{i2\theta}+e^{-i2\theta}+2\right) d\theta \qquad \text{by Rule 1 for exponentials} \\
&= \frac{1}{4}\left(\frac{1}{2i}e^{i2\theta}-\frac{1}{2i}e^{-2i\theta}+2\theta\right)+C \qquad \text{by Rule 4 for exponentials} \\
&= \frac{1}{4}\big(\sin(2\theta)+2\theta\big)+C \\
&= \frac{1}{4}\sin(2\theta)+\frac{1}{2}\theta+C.
\end{aligned}$$

**Exercise:** Show that

$$\int \cos 2\theta \sin 7\theta\, d\theta = \frac{1}{4i}\int (e^{i2\theta}+e^{-i2\theta})(e^{i7\theta}-e^{-i7\theta})\, d\theta,$$

and, proceeding as above, that the answer is $-\frac{1}{10}\cos(5\theta)-\frac{1}{18}\cos(9\theta)+C$.

*We moved to exponentials*
*used the Rule 4 for integration*
*moved back to trig functions.*

This shows the power of moving into the complex plane (and then out again).
The same process can be used for differentiations.

## 8.4 A slightly different approach

Looking at $e^{i\theta}=\cos\theta+i\sin\theta$ allows us to write

$$\cos\theta = \text{the real part of } e^{i\theta} = \mathrm{Re}\big(e^{i\theta}\big).$$

So sometimes we can work with $e^{i\theta}$ and "just take the real part at the end." This is a very common approach for **linear** problems in science and engineering where additions and differentiations are involved.

**Example:** Check that $y=e^{-2t}\cos(3t)$ satisfies the differential equation

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+13y=0.$$

We note that $e^{-2t}\cos(3t) = \mathrm{Re}\big(e^{(-2+3i)t}\big)$, so we substitute $y = e^{(-2+3i)t}$ and use Rule 3 for differentiating to get

$$\begin{aligned}\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y &= (-2+3i)^2 e^{(-2+3i)t} + 4(-2+3i)e^{(-2+3i)t} + 13e^{(-2+3i)t} \\ &= \Big((4-9-12i) + (-8+12i) + 13\Big)e^{(-2+3i)t} \\ &= \big(0+0i\big)e^{(-2+3i)t}.\end{aligned}$$

We see that $e^{(-2+3i)t}$ satisfies the differential equation.
Therefore $\mathrm{Re}\big(e^{(-2+3i)t}\big) = e^{-2t}\cos(3t)$ does satisfy the differential equation.

*Try the calculation with $y = e^{-2t}\cos(3t)$ to see how much more messy it is. Again, we begin to see the power of complex analysis.*

Incidentally, we see that $\mathrm{Im}\big(e^{(-2+3i)t}\big) = e^{-2t}\sin(3t)$ also satisfies that differential equation.

*Engineers and scientists often write oscillating quantities (currents in circuits, all sorts of waves)* as

$$\mathcal{A}e^{i(ft+\phi)}$$

where $\mathcal{A} =$ the amplitude, a real number
$ft =$ the phase variation and $\phi =$ phase constant,
$f$ is frequency, $t$ is time.

# 9 Polynomial Equations

We began by considering quadratic and cubic equations, and we now finish with some points about the general case

$$P_n(z) = z^n + c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_1 z + c_0 = 0,$$

where $c_i$'s are real numbers.

If $P_n(w) = 0$, we say that $z = w$ is a **root** or **zero** of the polynomial. This problem is widespread in maths, both ancient and modern. The early development of Algebra was greatly concerned with polynomials and their roots.

## 9.1 Formulas

We are all familiar with the ancient formula for the roots of $z^2 + c_1 z + c_0 = 0$ — we began the topic with it.

Tartaglia (1505–1557), who wrote probably the first book on artillery, found a formula for finding the roots of a cubic

$$z^3 + c_2 z^2 + c_1 z + c_0 = 0.$$

Cardano (1501–1576) stole the result and it is still given under his name in Mathematical handbooks.

He and Ferrari also found a formula for solving

$$z^4 + c_3 z^3 + c_2 z^2 + c_1 z + c_0 = 0.$$

The next challenge was obviously to find a formula for the roots of the quintic $P_5(z)$. Or better still, to find a formula for the roots of the general polynomial $P_n(z)$.

Many people took up the challenge but after 300 years a different kind of result emerged. Abel (1802–1829) and Galois (1811–1832) proved that the search was hopeless.

**There is no formula for the roots of $P_n(z)$ when $n \geq 5$**, except for special cases.

(For example we know $z^n - c = 0$ has solutions $z$ equal $c^{1/n}$ and we found a formula for the $n$th roots of a number, including a complex number, in Section 5.2.)

**This was a turning point in the history of mathematics**. Before this time, the standard approach to mathematical problems was to show how they worked by exhibiting the answer. In this case there was a more fundamental question: **Is there an answer**? Abel and Galois answered "no" and changed the course of mathematics.

Of course this did not say anything about the roots of $P_n(z)$, except that general formulas for them can only be found for $n = 1$, 2, 3 and 4.

## 9.2 A simpler question?

A more basic question is: Does every $P_n(z)$ actually have a root? And if so, how many are there all together?

(For example, it is obvious that $z^4 + z^2 + 3$ has no **real** roots because it is $\geq 3$ when $z$ is any real number, but it is not at all obvious that it has complex roots, nor how many.)

This is a tough question and the answer is known as the **Fundamental Theorem of Algebra**. The first satisfactory proof was given by Carl Gauss in his doctoral thesis in 1799.

> Every non-constant polynomial has at least one complex root.

Since it is easy to factorise a polynomial if we know one root, for example

because $z = 2$ is a root of $z^4 - z^3 - 8z + 8$

then $z^4 - z^3 - 8z + 8 = (z-2)(z^3 + z^2 + 2z - 4)$.

The theorem is readily used repeatedly to obtain:

> A polynomial of degree $n$ has at most $n$ distinct roots and they may all be written as complex numbers.
>
> We can write
>
> $$z^n + c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_1 z + c_0$$
>
> $$= (z - R_1)(z - R_2)(z - R_3)\ldots(z - R_n),$$
>
> where the complex numbers $R_1,\ R_2,\ \ldots R_n$ are the roots, which need not all be different.

There are some important observations.

1. This is one of the great **existence proofs** of mathematics. It does not tell you what the roots of, for example, $z^{91} + 68z^{16} - 41.38z + 1496$ are, but it does assure us that they exist and there are up to 91 possible different ones.

2. We can have repeated roots.

   For example the cubic $z^3 - 3z^2 + 4$ has roots $-1$, 2 and 2, so that it can be written

   $$z^3 - 3z^2 + 4 = (z-2)^2(z+1),$$

   with 2 a repeated root.

3. The theorem does tell us that all the roots can be expressed as complex numbers, so we have no more numbers to invent or discover.

4. In fact the theorem is true even if we allow the coefficients $(c_0, c_1, \ldots c_n)$ to be complex numbers.

   Of course we have seen a case of this when we solved $z^n - v = 0$ as $z = v^{1/n}$ and we saw that a complex number $v$ has $n$ different $n$th roots.

5. The development of algorithms for computing roots of polynomials to a given accuracy is an important part of **numerical analysis.**

## 9.3 Properties of roots of polynomials

A mathematical problem may involve three stages:

- checking that an answer does exist;
- finding properties of the answer;
- finding methods to calculate the answer in detail.

We have considered the first and third of these for the problem of finding the roots of a polynomial. We now show that complex-number theory produces a nice example of the second point.

There may be no general formula for solutions of $P_n(z) = 0$, but can we at least find out something **about** the roots?

**Example:** Here is something we can prove using the **complex-conjugate** operation.

Recall $z = a + ib$ has complex conjugate $\overline{z} = a - ib$

and the two simple properties:

$$\begin{aligned} \overline{(z_1 + z_2)} &= \overline{z}_1 + \overline{z}_2 \text{ or generally } \overline{(z_1 + z_2 + z_3 + \cdots + z_n)} = \overline{z_1} + \overline{z_2} + \overline{z_3} + \cdots + \overline{z_n}\,; \\ \overline{z_1 z_2} &= \overline{z}_1\, \overline{z}_2 \text{ or generally } \overline{(z_1 z_2 z_3 \cdots z_n)} = \overline{z_1}.\overline{z_2}.\overline{z_3} \ldots\ldots \overline{z_n}\,; \end{aligned}$$

i.e. for addition (and subtraction) and multiplication, you can take the complex conjugate first then do the other operations, or vice versa.

Now return to $P_n(z) = 0$ and suppose we have found $z = w$ as a root or solution, i.e. $P_n(w) = 0$ or

$$w^n + c_{n-1}w^{n-1} + c_{n-2}w^{n-2} + \cdots + c_1 w + c_0 = 0.$$

Take the complex conjugate of both sides:

$$\overline{(w^n + \cdots c_1 w + c_0)} = \overline{0} = 0.$$

Our rule about addition and complex conjugates means we get

$$\overline{w^n} + \overline{c_{n-1}w^{n-1}} + \cdots + \overline{c_1 w} + \overline{c_0} = 0.$$

Now we use our rule about products to find

$$\overline{w^n} + \overline{c_{n-1}}\,\overline{w^{n-1}} + \cdots + \overline{c_1}\,\overline{w} + \overline{c_0} = 0.$$

But all $c_i$ are real, so $\overline{c_i} = c_i$.

Also repeated use of the product rule, e.g.

$$\begin{aligned}\overline{w^3} &= \overline{w^2 w} \\ &= \overline{w^2}\,\overline{w} \\ &= \overline{w}\,\overline{w}\,\overline{w} \\ &= (\overline{w})^3,\end{aligned}$$

gives $\overline{w^m} = (\overline{w})^m$ for any $m$.

So finally

$$(\overline{w})^n + c_{n-1}(\overline{w})^{n-1} + \cdots + c_1\overline{w}_1 + c_0 = 0,$$

i.e. $P_n(\overline{w}) = 0$.

> If $w$ is a root of a polynomial that has real coefficients, then $\overline{w}$ is also a root, that is **roots come in complex-conjugate pairs**

This happened in our very first example, $z^2 - 2z + 2 = 0$, where the roots were $1 + i$ and $1 - i = \overline{1 + i}$.

## 9.4 Using all our knowledge

The above results sometimes allow us to cut down on the work and only seek certain solutions.

**Example:** If we know that

$$z^5 - 5z^4 + 9z^3 - 11z^2 + 8z - 6 = 0$$

has solutions $z = 3$, $i$ and $1 - i$, what are **all** the other solutions?

Because roots come in complex-conjugate pairs, we know immediately that $\overline{i} = -i$ and $\overline{1 - i} = 1 + i$ are also solutions.

That means we know 5 solutions and **there are no more**, because a 5th-degree polynomial can have at most 5 different roots according to the Fundamental Theorem of Algebra.

## 9.5 Roots of real numbers

Finding the $n$th roots of a real number $a$ is the same as solving $z^n - a = 0$, so the above theory can be used to get some neat results. For example, if an $n$th root of $a$ is complex, then the complex conjugate is also a root, i.e. the roots form complex-conjugate pairs.

To give another example, if the $n$th roots of $a$ are $w_1, w_2, \ldots, w_n$, then from Section 9.2 we know that

$$z^n - a = (z - w_1)(z - w_2)(z - w_3)\ldots(z - w_n).$$

That equation must be correct for all values of $z$, so we can put $z = 0$ to get

$$\begin{aligned} -a &= (-w_1)(-w_2)(-w_3)\ldots(-w_n) \\ &= (-1)^n w_1 w_2 w_3 \ldots w_n \end{aligned}$$

or

$$w_1 w_2 w_3 \ldots w_n = (-1)^{n-1} a.$$

Thus we have the nice result

The product of the $n$th roots of $a = (-1)^{n-1}a$.

These results also follow from the theory in Section 5.2, but not so easily and neatly as in the general proofs just given.

## 9.6 Summary

We now have a number of general results about polynomials and their roots:

- a polynomial of degree $n$ has at most $n$ distinct roots
- if the roots, including any repeated roots, are $R_1$, $R_2$, $\ldots R_n$, the polynomial may be written in terms of the factorization

$$(z - R_1)(z - R_2)(z - R_3)\ldots\ldots(z - R_n)$$

- if the coefficients defining the polynomial are real, the roots come in complex-conjugate pairs
- a formula derived using De Moivre's Theorem gives the $n$th roots of a complex number and hence the solutions for $z^n - w = 0$
- for any polynomial of degree 2, 3 or 4, there are formulas for the roots (available in various Mathematical handbooks), but there are no general formulas for the case where the degree is 5 or greater.

# 10 Conclusion

We have now found all the numbers we require in mathematics and have glimpsed the field of Complex Analysis, which turns out to be of major use in applications of mathematics to physical problems. We have also seen how important it is to have a variety of representations and to choose the most suitable one for any given problem.

Moving from linear equations to nonlinear polynomial equations has forced us to enlarge the set of numbers we use. These new developments feed back into linear algebra when we make use of complex eigenvalues and vectors. They lead us into complex-valued functions.

Nonlinear equations may involve complex numbers and can lead us into the new areas of chaos theory and fractals.

Finally, we note that the history of complex numbers provides a good example of the conceptual difficulties we all face when developing new mathematical ideas. It also shows the importance of using different representations, both algebraic and geometric, in mathematics.