# Population Modelling

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*Using graphics calculators* (Casio, Sharp and Texas Instruments calculators, unless otherwise stated) *for a variety of topics/activities.*

- Graphics-calculator activities for Years 9, 10 — written as part of the CQTP Program.
- Programs and program information for downloading (all three brands).
- *Using the TI-83*, *Using the CFX-9850*, *Using the EL-9650/9900*, — an introduction to the basic operations, suitable for Years 8–12.
- *Calculus* — basic commands and a variety of problems, suitable for Years 11, 12.
- *Sequences and Series* — basic commands and a variety of problems, suitable for Years 10–12.
- *Matrices* — suitable for Years 11, 12.
- *Coordinate Geometry* — basic commands and a variety of problems, suitable for Years 9, 10.
- *Complex Numbers* — suitable for Years 11, 12.
- *Programming a TI-83*, *Programming an EL-9650/9900* — suitable for teachers and keen students.
- *The Graphics Screen and Accuracy* — information to help you understand the graphical and numerical limitations of a graphics calculator.
- *Introduction to Complex Numbers* — complex numbers from the beginning, covering the basic operations, but set in the context of complex numbers as a mathematical structure.

# 1 Introduction

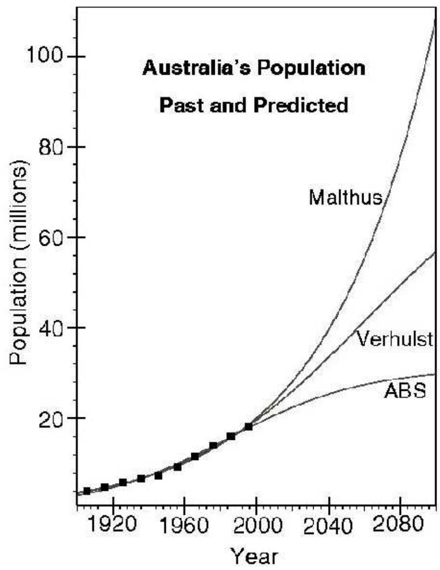
When mathematicians talk about playing with a model, chances are they don't mean a model plane or boat. They are probably talking about a mathematical model — a set of equations that describe in mathematics how a particular system works. There are mathematical models for many things, such as the planets revolving about the sun, heating iron ore in a blast furnace, pollution in a lake, how prices vary on the stock exchange, the spread of diseases and how populations (people, animals, bacteria, etc) change with time.

Population modelling started a long time ago, and one of the earliest modellers was Fibonacci (1170–1250). In his book *Liber abaci*, he modelled a rabbit population, starting with one pair of baby rabbits. If each adult pair of rabbits produces only one pair of baby rabbits each month, and if baby rabbits take one month to become adults, the numbers of pairs of rabbits in successive months are given by the famous Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, and so on. The next number is found by adding the previous two numbers. Fibonacci numbers are also found elsewhere in Nature. If you look at a pine cone, you will find the 'petals' spiral in two directions. The number of petals it takes to get once around is almost always a Fibonacci number. The same thing occurs in pineapples, sunflowers and many other flowers.

Much later, Thomas Malthus (1766–1834) in England startled the world by predicting that food would run out sometime in the future, because of the rapid increase in the human population. Based on the data he had at the time, Malthus predicted that the world population would increase exponentially, doubling every 40 years, thereby increasing at a faster and faster rate. (Forty years is the current doubling time of the world's population.) If you start with the number 1 and keep doubling it, you will see an example of exponential growth.

The models of Fibonacci, Malthus and some other scientists all predict that the population will grow faster and faster. This is an alarming prospect, but does not seem to happen in experiments performed when there are limited resources, such as food and space to live in. Experiments with small animals and fungi in the laboratory, and with larger animals in fenced areas in the field show that as the resources start to run out, the reproduction rate reduces and the rate of growth slows down. The Belgian scientist Pierre Verhulst (1804–1849) while at the Belgian military school, the Ecole Royale Militaire, developed a model, called the logistic model, which took into account these observations. He introduced the idea of a 'carrying capacity' or maximum sustainable population that an environment will support.

We can illustrate the Malthus exponential model and the Verhulst logistic model by looking at the population of Australia since 1900. The squares on the graph over the page show the Australian Bureau of Statistics figures for the number of people in Australia (in millions) up until 1996. If we model these data with an exponential curve (the Malthus model), we get the top curve in the figure. The middle curve is the Verhulst model. Both these curves fit the population numbers up to the present time well, but predict quite different future populations.




According to the Malthus exponential model, the population will continue to grow at a faster and faster rate, with a predicted population of about 109 million people in the year 2100, and about 587 million people in 2200. The Verhulst logistic model predicts that the population will keep on growing, but at a slower and slower rate; the predicted population in 2100 is about 57 million people, and the population would level off eventually at about 83 million people.

The Australian Bureau of Statistics uses a mathematical model to predict the population of Australia well into the future to assist in planning for the number of people who will be living here. The predictions of their model are shown as the bottom curve in the figure. It has the shape of a Verhulst curve, but levels out much faster than the middle curve, predicting a population in 2100 of about 30 million people, and a maximum population of about 31 million.

Prediction is one powerful aspect of a mathematical model. By putting in the numbers we know, such as for the Australian population, we can predict what a population will be in the future, according to our model. Of course, the accuracy of our predictions depends on how good our model is, that is how well it describes the phenomena that affect population growth.

Another important use of a population model is to predict what will happen to the population if something changes, for example if the birth rate drops, if the number of immigrants is decreased, or if, say in a war, many people die. Predicting changes in a population is particularly relevant to populations of animals, insects and plants which have become serious pests after being brought into Australia from overseas. These include rabbits, foxes, mice, cane toads and European carp among the animals, and prickly pear, Paterson's curse, salvinia, mimosa and scotch thistles, to name but a few of the plants. The populations of some of these have reached very high levels at times, causing serious problems for farmers and the environment.

How do we control such pests? Often there are a number of possible ways, but which one is best? Population models can be modified to include the effect of the release of a predator, the spread of a disease in the pest population, the effect of poisoning or some other control measure. It is then possible to use the models to predict what would happen to the population if the different control strategies were tried. The models can also be used to find the best way of carrying out a particular control measure. Sometimes the modelling is done together with small-scale experiments, but often only the mathematical model can be used because the experiments are too risky or too expensive.

In using a population model, we put the starting conditions and parameters (number of animals, how quickly they breed, etc) into our equations and predict the population at some later time. What if we change the starting conditions only slightly? We will end up with nearly the same final answer, right? Not necessarily. In some models, for example a variation on the Verhulst logistic model, with particular parameters, we find that the population does not change steadily towards some ultimate population, as we saw in modelling the Australian population, but changes rapidly and unpredictably with time. We say the model exhibits chaos: it loses its ability to predict, because a small change in the starting conditions produces a large change in how the population varies with time.

# 2 Population Problems

Mathematically, the problems here are about *iteration* and about *exponential processes*.

Iteration is the process of carrying out the same operation over and over again. Let's take a simple example, that of multiplying by 2. Start with the number 1. Multiply it by 2 to give 2. Multiply the answer 2 by 2 again to give 4. Multiply 4 by 2 to give 8, and so on.

If you have a calculator, you may be able to do many of the calculations in the problems just by pressing the [=] or [×] key. To do the calculation here, try this: press [1] [×] [2] [=], then just press [=] or [×] (depending on your calculator) to multiply by 2 each time. You'll have to keep count of how many times you have multiplied by 2. If this quick method doesn't work on your calculator, experiment to see what does.[1]

Iterating by multiplying by a constant (2 here) is an example of an *exponential process*. You may have heard the term exponential growth, which many people interpret to mean 'grow quickly'. But exponential growth has a precise mathematical meaning, and some interesting properties which we shall explore shortly.

Exponential iteration models a number of processes such as radioactive decay, population growth and absorption of light. If the constant we multiply by is larger than 1, we get growth; if it is less than 1 (but greater than 0), we get decay.

The use of scientific notation makes writing down our calculations much easier. For example, if we start with the number 5 and multiply it by 2 three times, we get $5 \times 2 \times 2 \times 2$, written as $5 \times 2^3$, which equals 40. $2^3$ means three 2s multiplied together. If we multiply 5 by 2 ten times, we have $5 \times 2^{10}$, which equals 5120. $2^{10}$ means ten 2s multiplied together. Some calculators have an exponentiation key, usually [$y^x$] or [$a^b$] or [∧], so we would press [2] [$y^x$] [3] [=] or [2] [$a^b$] [3] [=] or [2] [∧] [3] [ENTER] to calculate $2^3$.

From now on, [ENTER] can also mean [=] or [EXE].
Similarly, [∧] can also mean [$y^x$] or [$a^b$].

The calculator programs referred to in some of the problems are described in the last section.

[1] On a TI-83 and Sharp EL-9650/9900: [1] [ENTER] [×] [2] [ENTER] [ENTER] ....
On a Casio CFX-9850: [1] [EXE] [×] [2] [EXE] [EXE] ....

## 2.1 Exponential Iteration

Write down the results of the first 10 iterations of multiplying by 2, starting with 1.

## 2.2 Lots and Lots of Bacteria

Bacteria multiply (increase in number) by dividing — into two. One type of bacterium, *Streptococcus exponentiae*, divides every minute. If we start with 1 bacterium, it divides into 2 bacteria after 1 minute. Each of these 2 bacteria divides after 1 more minute, and so on. *The number of bacteria grows exponentially.*

Make up a table with time in the first column and the number of bacteria in the second.

*How many bacteria are there after 10 minutes? after 20 minutes? after 1 hour? after n minutes?*

*How long before there are 15,000 bacteria?*

*Why isn't the Earth covered metres deep in these bacteria?*

## 2.3 Malthus and Exponential Growth

Thomas Robert Malthus (1766 – 1834) made some worrying predictions for the world population, and his name is often associated with the idea of exponentially growing populations. Look up Malthus in an encyclopaedia or on the Internet to find out the details of his ideas. Why was he worried about the world's population?

Malthus looked at the United States population to try to verify his ideas. He concluded the growth was exponential. *From the numbers in the table below, can you tell if he was correct for the years until he died?*[2] *What about the population growth after about 1860?*

| Year | Population (millions) | Year | Population (millions) | Year | Population (millions) |
|---|---|---|---|---|---|
| 1790 | 3.9 | 1860 | 31.4 | 1930 | 122.8 |
| 1800 | 5.3 | 1870 | 38.6 | 1940 | 131.7 |
| 1810 | 7.2 | 1880 | 50.2 | 1950 | 150.7 |
| 1820 | 9.6 | 1890 | 62.9 | 1960 | 179.3 |
| 1830 | 12.9 | 1900 | 76.0 | 1970 | 203.2 |
| 1840 | 17.1 | 1910 | 92.0 | 1980 | 226.5 |
| 1850 | 23.2 | 1920 | 105.7 | 1990 | 248.7 |

[2] *Hint*: If the growth is exponential, each population should be a constant multiple of the previous value. Try a multiplier of 1.35, meaning the population increased by 35% every 10 years. The numbers you obtain only need to be close to the actual numbers, not exactly the same.

The following graphics-calculator programs are designed to make life easier for the teacher or to use on an overhead-projector calcualtor for the whole class to discuss. Your students could use the programs, but they may get more out of the lesson by doing some of the steps manually. One the programs are run, the data are stored in lists. These data can be copied to students' calculators for them to plot and fit.

The MALTHUS1 program (see Section 4) plots the first few data and fits an exponential function of the form $P(t) = ab^t$, where $a$ and $b$ are constants. You can do this manually on the calculator — the program just makes it easier. You might like to work out possible values of $a$ and $b$ manually using just the first two data points.

The MALTHUS2 program (Section 4) plots all the data and fits both an exponential function and a logistic function. It also shows extrapolation of the logistic curve to predict the US population in the future.

The AUSPOP and WORLDPOP programs (Section 4) plot the population of Australia and the world respectively and fit an exponential function and a logistic function. They also plot a logistic curve fitted to Australian/US Bureau of Statistics predictions to allow you to predict the populations in the future.

*Why might the populations not continue to increase exponentially?*

## 2.4 Cane Toads

The Hawaiian cane toad (*Bufo marinus*) was introduced into Australia to control sugar-cane beetles. From the original 101 toads released in north Queensland in June 1935, the population grew rapidly and spread across the countryside. The table below shows the total land area of Australia colonised by cane toads for the years 1939 to 1974.

| Year | Area (1000 $km^2$) | Year | Area (1000 $km^2$) |
|---|---|---|---|
| 1939 | 33.8 | 1959 | 202 |
| 1944 | 55.8 | 1964 | 257 |
| 1949 | 73.6 | 1969 | 301 |
| 1954 | 138 | 1974 | 584 |

*Is exponential growth a good model here?* You can get a rough idea by the process we used for the Malthus data — finding ratios of successive values — but a plot of the data together with an exponential fit (graphics calculator) will provide a better answer.

Given that the area of Queensland is 1728 thousand $km^2$ and the area of Australia is 7619 thousand $km^2$ when, according to the exponential model, will (did) the cane toads colonise all of Queensland? all of Australia?

The cane growers were warned by Walter Froggart, president of the New South Wales Naturalist Society, that the introduction of cane toads was not a good idea and that the toads would eat or poison the native ground fauna. He was immediately denounced as an ignorant meddlesome crank. He was also dead right.

## 2.5 Rabbits and Fibonacci Numbers

Fibonacci (fib–on–archie), real name Leonardo Pisano (Leonardo of Pisa), was born in about 1170. He too thought about populations, but much earlier than Malthus. One of his problems concerning a rabbit population led to the famous Fibonacci numbers

$$1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad \ldots.$$

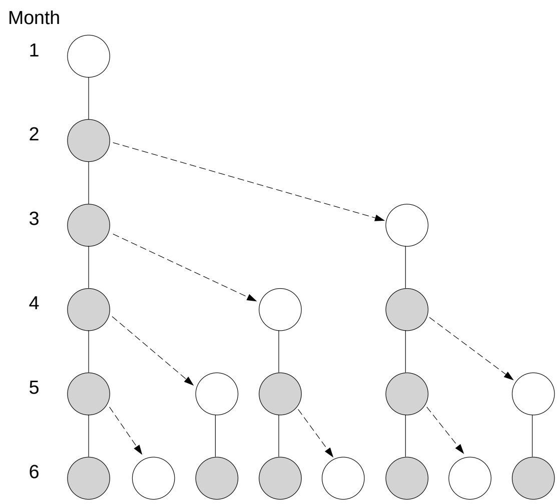
*The Rule*: Add the last 2 numbers to get the next number.

*Write down the first 20 Fibonacci numbers.*

Look up Fibonacci numbers and how they relate to the 'Golden Number' or 'Golden Ratio' or 'Golden Section' (architecture). Fibonacci numbers also turn up in Nature. See if you can find out where.

Here is Fibonacci's rabbit problem. See if you can understand why the Fibonacci numbers give the number of *pairs* of rabbits each month and answer the question. The diagram[3] might help.

> A pair of new-born rabbits is put in a pen. *How many pairs of rabbits are there after a year if, every month, each adult pair produces a new pair?* The rabbits become adult one month after birth.




In the diagram, an open circle represents a *pair* of immature rabbits (too young to breed) and a shaded circle a *pair* of mature breeding rabbits. The arrows lead to offspring. Adding the number of circles for each month gives the number of pairs of rabbits — the Fibonacci numbers.

Note that the growth here is no longer exponential (we are not multiplying by a constant to obtain the next number), but the number of rabbits still increases rapidly.

[3]Slightly modified from one by David Schweizer at *maths.holycross.edu/~davids/fibonacci/course.html* Lecture 1. Used with the author's permission.

## 2.6 The Discrete Logistic Model — Fewer Bacteria

Here we use a version of the Verhulst model, called the discrete logistic model, to predict the growth of a population of bacteria. The model is in the form of a difference equation which tells you how to calculate the population in an hour's time if you know the population now:

$$P_{n+1} = A \times P_n \times (1 - P_n).$$

In this model, $P_n$ is a measure of the population (in millions of bacteria) at the end of the $n$th hour and $A$ is a number that depends on how fast the bacteria reproduce. For our calculations, we take $A = 2$ and the starting population $P_0 = 0.1$.

To calculate $P_1$, the population after 1 hour, put $n = 0$ and $P_0 = 0.1$ in the equation:

$$P_1 = 2 \times P_0 \times (1 - P_0) = 2 \times 0.1 \times (1 - 0.1) = 0.18.$$

To calculate $P_2$, the population after the second hour, put $n = 1$ in the equation:

$$P_2 = 2 \times P_1 \times (1 - P_1) = 2 \times 0.18 \times (1 - 0.18) = 0.2952,$$

and so on. After a few more steps (hours), you should find the population stabilises at a particular number. *What is the number?*

To speed up this process, on a calculator, type ( [ENTER] ≡ [EXE] )

0.1 → [ALPHA] P [ENTER]

2P(1−P) → P [ENTER]

This will give you the next value for P. If you now keep pressing [ENTER], the calculator will repeatedly execute the last line to give successive values for P.

Next let $A = 3.2$ and keep $P_0 = 0.1$:

0.1 → [ALPHA] P [ENTER]

3.2P(1−P) → P [ENTER]

and keep pressing [ENTER].

You'll need to run the population for about 18 hours this time before it settles down. What happens here? *Draw a plot of population versus time.*

Now try $A = 3.8$ and $P_0 = 0.1$. This one is weird! The population varies wildly between 0.1 and 1, with no hope of prediction. *Plot this one too.* You've discovered chaos (the mathematical version).

*What happens with other values of $A$ and $P_0$?*

The sequence grapher on a graphics calculator can be used to graph values of $P_n$ vs $n$. The LOGISTIC program (Section 4) sets this up for you for the bacteria here and for the kangaroos in the next question.

## 2.7 Kangaroo Management

Part of an ADFA Maths Lab adapted from *Stimulating Mathematical Interest with Dynamical Systems* by M.B. Durkin, The Maths Teacher 89, 242–24 (1996).

In late 2005 you are hired by the State Forestry Department, with your main task to assist in the management of the kangaroo population in a remote forest called Hamt Reserve. The possibility of culling of kangaroos in the reserve is under consideration.

The kangaroo population in the reserve is given by a difference equation

$$P_{n+1} = 1.8P_n - 0.8(P_n)^2, \qquad (1)$$

where $P_n$ is the number of kangaroos in the reserve at the end of year $n$ in tens of thousands, i.e. *one unit of $P$ equals 10,000 kangaroos.* At the end of 2005, there are 8000 kangaroos in the reserve ($P_0 = 0.8$).

### The First Task

As a training exercise, management asks you to model and report on a scenario containing several events that would affect the kangaroo population.

*Write a short report on the outcome of the following scenario. The report should include a mathematical analysis with calculations, tables and/or graphs to substantiate your conclusions.*

### The Scenario

- If there were no natural disasters in 2006, what would the kangaroo population be at the end of 2006? Do this and the following calculations manually (without a program) using equation (1).[4]
- Unfortunately, at the end of 2006, there is a short but fatal outbreak of the dreaded rootoxis which kills around 4000 kangaroos. What is the population of kangaroos at the end of 2007? When will the kangaroo population recover to more than 9000 kangaroos if there are no more natural disasters?
- Following the rootoxis epidemic, on Christmas Day 2008 there is a forest fire in a nearby forest which results in 2000 kangaroos from that forest migrating into Hamt Reserve. What will the population of kangaroos in Hamt Reserve be at the end of 2009?
- After these two events, there are no more natural disasters. What will the kangaroo population be after a long time? The number here is the limiting capacity or maximum sustainable population of the reserve.

[4] *Calculator hint*: Store the initial population in memory P and repeatedly execute the calculation $1.8\text{P} - 0.8\text{P}^2 \rightarrow \text{P}$ by pressing ENTER / EXE the required number of times. Make sure you understand why this works.

**Effect of Culling**

Impressed by your previous report, management has now put you in charge of undertaking a feasibility study into whether culling of kangaroos is necessary/desirable in Hamt Reserve. Your analysis will be a crucial factor in the decision-making process.

*Write a report addressing the following questions. Again, a mathematical analysis including calculations, tables and/or graphs is required to substantiate your conclusions. Add an executive summary for your boss, summarising your findings and making suitable recommendations.*

**1.** What is the modified form of equation (1) if $H$ kangaroo units are culled each year?

We assume here, for simplicity, that all the kangaroos are killed close to the end of the year, otherwise the killing of the female kangaroos in particular would affect the number of births and deaths, and consequently the growth rate.

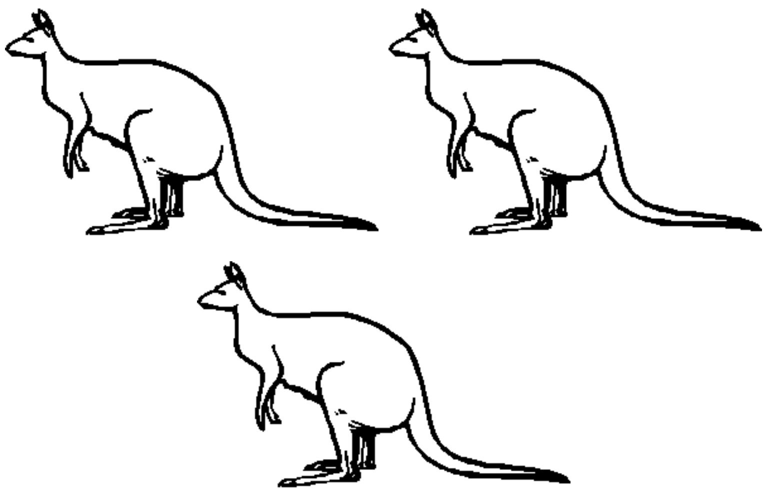
**2.** What would happen if 720 kangaroos were culled each year ($H = 0.072$), a value used in a nearby reserve? Assume the initial population is that given above for the year 2005, $P_0 = 0.8$. What is the long-term population?

What if the initial population were $P_0 = 0.3$? $P_0 = 0.095$?

**3.** What would happen if 2400 kangaroos were culled each year ($H = 0.24$)? Assume again that $P_0 = 0.8$. What is the long-term population?

What if the initial population were $P_0 = 1$? $P_0 = 1.5$?

**4.** What about $H = 0.2$? It turns out (experiment and see — the LOGISTIC program might help here) that this is the largest number of kangaroos which could be culled annually without the kangaroos dying out in Hamt Reserve. Note that the initial population must be larger than 0.5. What is the long-term population in this case?



Kangaroos from http://kwest.net/desk-top_publishing/graphics/animals/KANGAROO.GIF. Accessed on 18/9/2002.

## 2.8 Population-Projection Matrices

### Leslie matrices

We divide a population into a number of classes — here we shall assume three classes, referring to three age groups: young $y$, adults $a$ and seniors $s$. The population is then described by the vector

$$\boldsymbol{v} = \begin{bmatrix} y \\ a \\ s \end{bmatrix}.$$

A $3 \times 3$ transition matrix $\boldsymbol{T}$ tells us how the population evolves. For example, if the population to start with is

$$\boldsymbol{v}_0 = \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix},$$

after one cycle it is

$$\boldsymbol{v}_1 = \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix} = \boldsymbol{T}\boldsymbol{v}_0.$$

In problems leading to a Leslie transition matrix,

- in each cycle, members of the other classes produce a certain number of new young in class 1,
- a certain fraction of each class survives to move into the next class; the rest die,
- all members of the top class die.

This leads to a Leslie matrix $\boldsymbol{T}$ that is zero everywhere except possibly

- along the top row after the first element — the birth rates for each class
- in the elements along the diagonal parallel to and just below the main diagonal — the survival rates for each class.

$$\begin{bmatrix} 0 & * & * & * & * & \cdots & * & * \\ * & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & * & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & * & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & * & 0 \end{bmatrix}$$

Leslie discovered these matrices in the 1940's when he pioneered this way of exploring how populations can develop. He taught himself matrix algebra while he was in hospital with TB.

### Leslie matrices and beetles

**(a)** For a beetle population: during each cycle, each adult produces on average 2.75 young and each senior produces on average 2.5 young; one quarter of the young survive to become adults and one half of the adults survive to become seniors. In a Leslie-matrix problem, all the seniors die.

Find the Leslie transition matrix $\boldsymbol{T}$.

*Good strategy*: Write out the linear equations for $y_1$, $a_1$, $s_1$ in terms of $y_0$, $a_0$, $s_0$ and convert to matrix form.

**(b)** If we start with 40 young and no adults or seniors, show that after one cycle

$$\boldsymbol{v}_1 = \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}.$$

You should have entered $\boldsymbol{T}$ into [A], the initial $\boldsymbol{v}$ into [B] and evaluated [A][B].

**(c)** Multiply repeatedly by $\boldsymbol{T}$ (the program POP (Section 4) helps here), and record $\boldsymbol{v}$ and the total population $P = y + a + s$ after 11, 12 and 13 cycles. What happens to the total population? to the ratios of the numbers in the different classes?

### Populations and oscillations

Workers other than Leslie had independently used matrix algebra in population models. The first was Harro Bernardelli, who published a paper in 1941 in the Journal of the Burma Research Society with the title *Population Waves*. Bernardelli's paper was unusual in focussing not on the eventual stability of the population structure, but on intrinsic oscillations in the population structure. He had observed oscillations in the age structure of the Burmese population between 1901 and 1931. As an abstract model for such oscillations, he proposed a matrix model for the evolution of the population with

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

and showed by numerical calculations that this gave rise to apparently permanent oscillations in the age structure.

**(a)** Use POP, with the vector $\boldsymbol{v}$ set initially to $\begin{bmatrix} 1 \\ 0.01 \\ 0.01 \end{bmatrix}$ (population in three age groupings in millions) and Bernardelli's matrix $\boldsymbol{T}$, for 12 cycles, recording the total population $P$ at each cycle. Plot $P$ versus cycle number, joining up the points with straight lines. Discuss your findings.

**(b)** Repeat (a) using $\boldsymbol{T} = \begin{bmatrix} 0 & 0 & 5 \\ 0.7 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$. Describe your results in words.

## Killer Whales

*From a Year-12 project set by Margaret MacLaughlan, St Francis Xavier College.*

Leslie matrices are just a special case of a more general population-projection matrix. In the more general case, animals in a class may remain in that class for more than 1 cycle. The probability that an animal remains in a particular class for any given cycle is an element on the diagonal of the matrix, immediately above the entry in the Leslie matrix, which gives the probability of moving to the next class in any given cycle.

Therefore we have a matrix whose elements along the top row give the fecundity or birth rate per animal per cycle for each class, whose diagonal elements give the probability of an animal remaining in a particular class in any cycle and whose elements below the diagonal give the probability of an animal moving to the next class in any cycle. The fact that the latter two numbers in any column do not add up to 1 means that some animals in each class die each cycle.

For female killer whales, we have 4 classes — yearlings (individuals in the first year of life), juveniles (past the first year, but not mature), mature females and post-reproductive females. The mean period in the juvenile stage is 13.4 years and in the mature stage 22.1 years, with an overall lifetime of 80 – 90 years. Details in Brault and Caswell, *Pod-specific demography of killer whales*, Ecology 74, 1444–1454 (1993).

The population-projection matrix for female killer whales is given below. The time for one cycle is one year.

$$\boldsymbol{T} = \begin{bmatrix} 0 & 0.0043 & 0.1138 & 0 \\ .9775 & 0.9111 & 0 & 0 \\ 0 & .0736 & 0.9534 & 0 \\ 0 & 0 & 0.0452 & 0.9804 \end{bmatrix}$$

**1.** What happens to the total population of female killer whales over time according to this model if initially there are

**(a)** 50 juveniles and 50 mature females?

**(b)** 30 juveniles, 40 mature females and 30 post-reproductive females?

**2.** Does the trend of the total population depend on the starting values?

*Hint*: If $\boldsymbol{T}$ is in matrix [A] and the initial population $\boldsymbol{v}_0$ is in the $4 \times 1$ matrix [B], executing the command [A] [B] → [B] (mat A mat B → mat B) and repeatedly pressing ENTER / EXE will generate successive population vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, etc.[5] The symbol → stands for the STO key.

**PTO**

[5] You could also change the last line of POP to [B](1,1) + [B](2,1) + [B](3,1) + [B](4,1) and repeatedly run POP by pressing ENTER / EXE . This will give you the total population directly.

**3.** The value of 0.1138 in the top row of $\boldsymbol{T}$ gives the number of live births per mature female per cycle (year). To what value could this birth rate fall before the total population starts to decrease?

This birth rate is clearly important for the overall survival of killer whales.

**4.** How sensitive is the population to the survival rates of yearlings (0.9775), juveniles (0.9111), adults (0.9534) and post-reproductive females (0.9804)?

You might like to quantify your answers here by determining what percentage decrease in each rate is needed to stop the population growing.

*Do your answers make sense?*

## 2.9 Solutions

### Exponential Iteration

$1 \times 2 = 2 = 2^1$
$1 \times 2 \times 2 = 4 = 2^2$
$1 \times 2 \times 2 \times 2 = 8 = 2^3$
$1 \times 2 \times 2 \times 2 \times 2 = 16 = 2^4$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 = 32 = 2^5$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64 = 2^6$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 128 = 2^7$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256 = 2^8$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 512 = 2^9$
$1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1024 = 2^{10}$

### Lots and Lots of Bacteria

You should end up with a table containing the numbers above and lots more. From your table, you can read off the answers to the questions. If you are using a calculator, it will probably switch to scientific notation when the numbers become large enough; powers of 10 are just like powers of 2, but much easier to write down. $10^3$ is a 1 followed by three zeros, $10^{10}$ a 1 followed by 10 zeros, and so on.

| **Time** | **Number of Bacteria** |
|---|---|
| after 1 minute | 2 |
| after 10 minutes | 1024 ($= 2^{10}$) |
| after 20 minutes | 1048576 ($= 2^{20}$) |
| after 1 hour | $1.152921505 \times 10^{18}$ ($= 2^{60}$) |

The number of bacteria after $n$ minutes is $2^n$.
It takes 14 minutes before there are 15,000 bacteria.

The Earth isn't covered with these bacteria because in reality the growth is not exponential but has a limit: environmental conditions ultimately limit growth and scientists continue to discover new antibiotics. See Section 2.6.

### Malthus and Exponential Growth

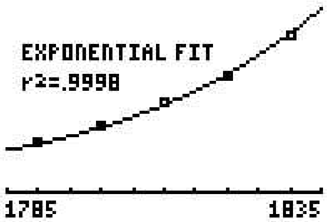
Thomas Malthus was worried about the world's population because he believed that population growth increased exponentially such as in the examples above, but that the food supply would only grow linearly (as a straight line — much more slowly). In other words, left unchecked, the human population would eventually exceed its food and other resources, leading to overcrowding, poverty, malnutrition, disease, crime and war. You can find out more about Malthus in most encyclopedias or from *www.igc.org/desip/malthus*.

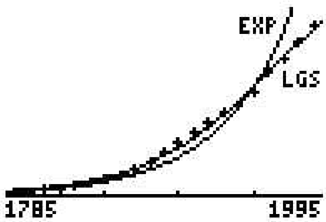
Starting at 1790, multiplying each number in the table by 1.35 gives a number close to the next number, up until 1860 — the US population increased by about 35% every ten years from 1790 to 1860. The growth rate then decreased to approximately 23%, 29%, 24%, and then slowly down to $9\frac{1}{2}$% in the decade to 1990. A marked decline in growth occurred between 1910–20 and 1930–40, during the two World Wars. The growth rate picked up a bit after the wars (the baby boom: 1950–60) and then slowly declined again.

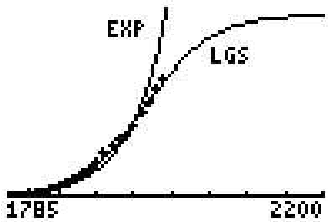
In the left-hand figure below, we see an exponential fit to the US population for the years 1790–1840. The fit is excellent, as shown by the coefficient of determination.

The middle figure shows exponential and logistic fits to the full data set 1785–1990. The exponential fit ($r^2 = 0.9676$) is not so good now and doesn't seem to follow the trend in later years. The logistic fit is much better.

The right-hand figure shows an extrapolation of the two curves to the year 2200. The exponential model predicts a US population in 2200 of more than 29 billion people. The logistic model predicts a population of about 386 million people in 2200, eventually stabilising at about 388 million people.



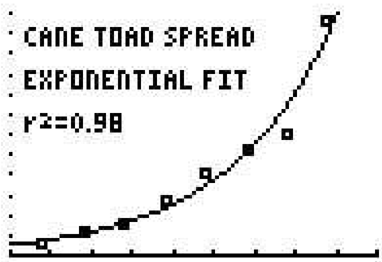




### Cane Toads

The ratio of successive terms jumps about a bit, between 1.17 and 1.94, with a mean of 1.34. The population is therefore increasing roughly exponentially.

The exponential fit to the data looks reasonable, with the value from 1969 a bit low (a drought period from 1964 to 1969?). The curve of best fit is $y = 36.91 \times 1.080^t$ or $y = 36.91e^{0.0774t}$, where $t$ is years since 1939.




Using a graphics calculator to solve $y = 1728$ and $y = 7619$, we find that, according to the exponential model, Queensland was overrun by cane toads between 1988 and 1989, and that Australia will be overrun between 2007 and 2008. Clearly and fortunately, there are some factors that restrict the spread of the cane toad.

## Rabbits and Fibonacci Numbers

The first 20 Fibonacci numbers are

1 1 2 3 5 8 13 21 34 55 89 144 233 377 610 987 1597 2584 4181 6765.

### The Golden Number/Ratio/Section

The Golden Ratio (also called the Divine Proportion) is denoted by the Greek letter $\phi$ or $\tau$. If a length is divided in the Golden Ratio $1 : \phi$, the ratio of the longer part to the whole, $\phi : 1+\phi$, is also the Golden Ratio. When used to construct a rectangle, this ratio was thought to make the rectangle pleasing to the eye. Hence the Golden Ratio occurs everywhere, and has been used to design buildings from the Parthenon to the United Nations building in New York, as well as by artists and musicians. It is closely connected with the Fibonacci series and has a value of $(1+\sqrt{5})/2 = 1.61803\ldots$.

Fibonacci numbers are evident in particular (equi-angular or logarithmic) spirals which appear frequently throughout the natural world — in things as small as the double helix and other microscopic twisting structures, to the galaxies that move in equi-angular spirals. These spirals also account for gastropods growing while maintaining their shape. In order for all leaves on a stem to catch sunlight, they are arranged in equiangular spirals that incorporate the Fibonacci numbers. Other examples of this natural phenomenon include pine-cone seeds, flower petals, sunflower seeds, the horns of mountain goats, elephant tusks and lions' claws, scales in pineapples, and so on. Find out more from encyclopedias or from *www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html*.

See also Peter Enge's articles and problems (highly recommended) on the Golden Ratio in CIRCUIT on the CMA web page *www.canberramaths.org.au*.

### How many pairs will there be in one year?

1. There is only one pair of immature rabbits during the first month.
2. At the beginning of the second month they mate, but there is still only 1 pair during the month.
3. At the beginning of the third month the female produces a new pair, so now there are 2 pairs of rabbits in the pen.
4. At the beginning of the fourth month, the original female produces a second pair, making 3 pairs in all in the pen.
5. At the beginning of the fifth month, the original female has produced yet another new pair, while the female born two months ago produces her first pair, making 5 pairs in all.

   and so on.

The number of pairs of rabbits after a year is the 12th Fibonacci number, or 144 pairs.

Why do the Fibonacci numbers appear as the number of rabbits in the pen each month?

If we let $f(n)$ be the number of pairs of rabbits in the pen at the start of the $n$th month, we will show that $f(1) = 1$, $f(2) = 1$ and $f(n) = f(n-1) + f(n-2)$, which is exactly the definition of the Fibonacci numbers.

We start in month 1 with one newly born pair, so $f(1) = 1$.

There is also only 1 pair during month 2, since although the adults mate at the start of month 2, babies are not born until the start of month 3: $f(2) = 1$.

Now look at the $n$th month.

All the rabbits from the previous month ($f(n-1)$ pairs of them) survive. Any rabbit (pair) that was alive 2 months ago is now able to produce a new pair; we assume they produce 1 and only 1 new *pair* per month. Thus, the number of newly born pairs is the same as the number of pairs alive 2 months ago, $f(n-2)$. The total number of rabbits this month is the sum of all the rabbits alive last month and those that are newly born this month, that is $f(n) = f(n-1)+f(n-2)$ — the definition of the Fibonacci numbers.

**Kangaroo Management**

The calculation of the population $P_n$ can be done manually on a calculator (following the calculator hint in the question) or by using the built-in sequence grapher, depending on the sophistication of your students and the type of calculator they have. The LOGISTIC program (Section 4) sets up the sequence grapher for the problem here.[6]

We have the difference equation for the kangaroo population $P_{n+1} = 1.8P_n - 0.8(P_n)^2$, with $P_0 = 0.8$ corresponding to the (end of) year 2005.

Using this and incorporating the rootoxis outbreak in 2006 by subtracting 0.4 (4000 kangaroos) from the 2006 population, we have the following number of kangaroos in subsequent years.

| **Year** | $n$ | $P_n$ | **Number of Kangaroos** |
|---|---|---|---|
| 2005 | 0 | 0.8 | 8000 |
| 2006 | 1 | $0.928 - 0.4 = 0.528$ | 5280 |
| 2007 | 2 | 0.7274 | 7274 |
| 2008 | 3 | 0.8860 | 8860 |
| 2009 | 4 | 0.9668 | 9668 |

The number of kangaroos has recovered to 9668 by the end of the year 2009.

[6] The sequence graphers on the TI and Sharp calculators write $P_n$ in terms of $P_{n-1}$, so it is necessary to rewrite the difference equation as $P_n = 1.8P_{n-1} - 0.8(P_{n-1})^2$ if you use this method.

If we include the migration of 2000 kangaroos at the end of 2008, we have the following numbers.

| Year | $n$ | $P_n$ | Number of Kangaroos |
|---|---|---|---|
| 2008 | 3 | 0.8860 + 0.2 = 1.0860 | 10,860 |
| 2009 | 4 | 1.0113 | 10,113 |
| 2010 | 5 | 1.0022 | 10,022 |
| 2011 | 6 | 1.0004 | 10,004 |
| 2012 | 7 | 1.0001 | 10,001 |
| 2013 | 8 | 1.0000 | 10,000 |

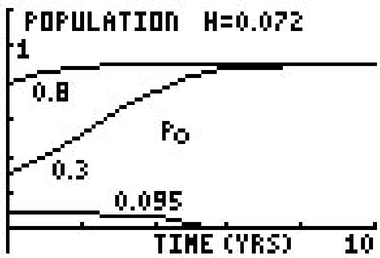
The population in the reserve at the end of 2009 would be 10,113. In subsequent years, the population declines to the equilibrium population of 10,000, the population after a long time.

Effect of Culling

1. If $H$ kangaroo units are killed each year, this number is subtracted from the value for $P_{n+1}$ that we calculated above, giving the difference equation

$$P_{n+1} = 1.8P_n - 0.8(P_n)^2 - H.$$

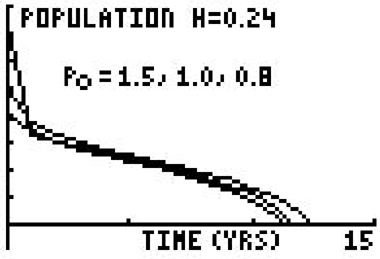
2. With $H = 0.072$ and an initial population of 0.8 units, the long-term population would be 0.9 units or 9000 kangaroos.



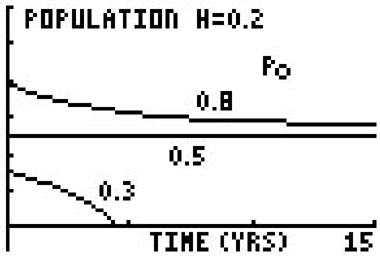

We find that[7] if the initial population is greater than 0.1 kangaroo units, the population will tend toward a stable value of 0.9. If the initial population is less than 0.1 kangaroo units, the population will tend to 0.

[7]Theory helps a lot here, but you can reach the same conclusions by experimenting with numbers on your calculator. Using the LOGISTIC program may help with this.

**3.** With $H = 0.24$, the population would die out, no matter what the initial population.




**4.** With $H = 0.2$, the long-term population would be 0.5 units or 5000 kangaroos, the maximum sustainable population with this level of hunting, provided that the initial population is greater than 5000. If the initial population is less than 5000, the population will die out.




If this level of hunting were chosen, any natural disaster that killed more than a few kangaroos after the population had levelled off at 5000 would bring the population to below 5000, and it would therefore die out. There is no margin for error with this level of hunting. In practice, a smaller value than 2000 would be chosen for the number of kangaroos killed annually, thereby leaving a margin to allow for natural disasters.

## Population-Projection Matrices

### Leslie matrices and beetles

**(a)** Writing out the equations for the three age classes,

$$\begin{aligned} y_1 &= 0y_0 + 2.75a_0 + 2.5s_0 \\ a_1 &= 0.25y_0 + 0a_0 + 0s_0 \\ s_1 &= 0y_0 + 0.5a_0 + 0s_0 \end{aligned} \quad \text{or} \quad \begin{bmatrix} y_1 \\ a_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ a_0 \\ s_0 \end{bmatrix}.$$

Therefore, $\boldsymbol{T} = \begin{bmatrix} 0 & 2.75 & 2.5 \\ 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$.

**(b)** $\boldsymbol{T}\begin{bmatrix} 40 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$.

**(c)**

| After cycle | $\boldsymbol{v}$ | Total Pop'n |
|---|---|---|
| 1 | $\begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}$ | 10 |
| 2 | $\begin{bmatrix} 27.5 \\ 0 \\ 5 \end{bmatrix}$ | 32.5 |
| ⋮ | ⋮ | ⋮ |
| 11 | $\begin{bmatrix} 17.320 \\ 4.305 \\ 2.184 \end{bmatrix}$ | 23.809 |
| 12 | $\begin{bmatrix} 17.299 \\ 4.330 \\ 2.153 \end{bmatrix}$ | 23.781 |
| 13 | $\begin{bmatrix} 17.289 \\ 4.325 \\ 2.165 \end{bmatrix}$ | 23.779 |

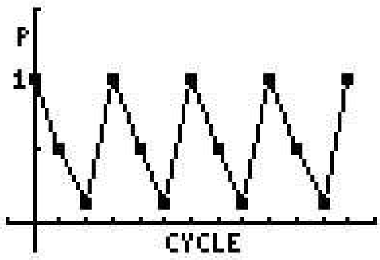
The total population seems to be stabilising at a little under 24 and the ratios of the populations in the 3 classes at about 8 : 2 : 1.

Populations and oscillations

**(a)** After 1 cycle: $\begin{bmatrix} 0 & 0 & 8 \\ .5 & 0 & 0 \\ 0 & .25 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ .01 \\ .01 \end{bmatrix} = \begin{bmatrix} .08 \\ .5 \\ .0025 \end{bmatrix}$, so $P = 0.5825$.

From successive cycles, we build up a table.

| **Cycle** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ⋯ |
|---|---|---|---|---|---|---|---|---|
| **Pop'n** | 1.02 | 0.5825 | 0.185 | 1.02 | 0.5825 | 0.185 | 1.02 | ⋯ |

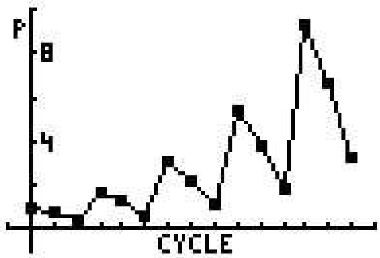



The population is oscillating or going in waves, with no overall growth or decline.

A group of young must first become adults (with a survival rate of 0.5) and then seniors (with a survival rate of 0.25) before producing a new group of young; the process then repeats itself. The overall survival rate between young and seniors of $0.5 \times 0.25 = 1/8$ is balanced by a birth rate of 8, so that the overall population is not growing or declining.

**(b)** The populations are now 1.02, 0.755, 0.41, 1.785, 1.321, 0.718, 3.124, 2.312, 1.256, 5.467, 4.046, 2.197, 9.566, 7.081, 3.846, 16,741, ....

The population is oscillating, but growing overall. The birth rate of 5 more than compensates for the death rate of $1 - 0.7 \times 0.5 = 65/100$.

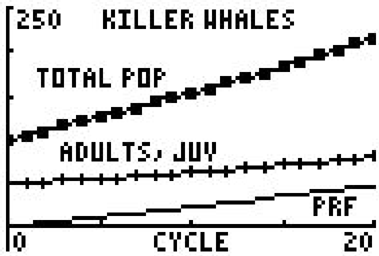


### Killer whales

Teachers (and students) may find the POPMATX4 program (TI-83 family only — Section 4) useful for experimenting with different values in the population-projection matrix.

The long-term total population varies as $\lambda_m^t$, where $\lambda_m$ is the largest eigenvalue of the population-projection matrix and $t$ is time. Therefore, if $\lambda_m > 1$, the population increases; if $\lambda_m = 1$, the population is steady; if $\lambda_m < 1$, the population decreases. The value of $\lambda_m$ is displayed on the second plot produced by POPMATX4, that of total population versus cycle.

In the two cases of different initial populations, we see the same trends: the overall female population increases, as do the populations in each of the classes.




After a reasonable number of cycles, there are approximately equal numbers of juveniles, mature adults and post-reproductive females (PRF), with about one tenth of that number of yearlings (not shown).[8] The fact of an overall increase in population and the eventual ratios of the populations in the four classes does not depend on the initial populations in each of the classes.

Experimenting with different values of the birth rate for mature females, either manually or using POPMATX4, shows that the population becomes steady when the value is about 0.055 ($\lambda_m \approx 1$), i.e. about 48% of the observed value. For birth rates less than 0.055, the total population will decline.

The female population is more sensitive to the survival rates, especially of the mature adults. Reducing the value of 0.9534 for mature adults to 0.905 (a reduction of only 5%) is enough to stop the population growing. For the other classes, the corresponding values are: juveniles 0.9111 down to 0.82 (10%); yearlings 0.9775 down to 0.49 (50%). The growth or otherwise of the population is unaffected by the survival rate of post-reproductive females.

[8]These ratios are those of the components of the eigenvector of the population-projection matrix corresponding to $\lambda_m$.

# 3 Other Exponential Problems

## 3.1 Piles of Paper

A ream of paper (500 sheets) is about 50 mm thick, so that one sheet is about 0.1 mm thick. Take a sheet of paper, cut it in half and put the two halves one on top of the other. Cut this pile of 2 pieces in half and make a pile of 4 pieces. Keep cutting the pile in half and stacking the pieces up.

Now suppose you could make 42 cuts altogether (you'd need big scissors!). *How high would your final pile be?* Try making up a table like the one below to keep track of your pile. *Where would your pile reach to?* Kilometres might be a good unit to use eventually.

| Cut | Height of pile | |
|---|---|---|
| number | in sheets | in mm |
| 1 | 2 | 0.2 |
| 2 | 4 | 0.4 |
| 3 | 8 | 0.8 |
| 4 | 16 | 1.6 |

This is an example of where maths lets you find an answer to something you can't actually do in real life.

## 3.2 Shoeing a Horse

A rich man sends his horse to the blacksmith to have 4 new horseshoes put on. Each shoe needs 5 nails. The blacksmith offers to charge either $100 per nail (they're gold!), or 1c for the first nail, 2c for the second nail, 4c for the third nail, and so on, the cost doubling each nail. *Which offer should the rich man take?*

Think first which offer *you* would take. Then do some calculations. Don't forget to add up the total cost at the end. A calculator might be useful. *Did you pick the better offer? Was there much difference?*

## 3.3 Interest Rates

*There is lots of material on interest rates – a short introduction is given here to place it in the context of exponential problems.*

Once you have some money in the bank, you start to think about interest, and you might want to answer a question like the one below to work out how much money you will have some time in the future:

> *If the annual interest rate on a bank account is 12% compounded monthly and you deposit \$10, how much money will you have after 1 year? after 5 years? after 10 years?*

What does this mean? In simpler terms, it means that every month the bank will pay you an amount of interest equal to 1% (an annual interest rate of 12% means a monthly interest rate of 12%/12 = 1%) of the amount you have in the account at the end of the month. So, after the first month, the bank will pay into your account 1% of \$10 or

$$0.01 \times \$10 = \$0.10 = 10\text{c}$$

in interest, and you will then have

$$10 + 0.10 = 1.01 \times \$10 = \$10.10$$

in your account.

After the second month, the interest will be 1% of \$10.10 or

$$0.01 \times \$10.10 = \$0.101 = 10.1\text{c},$$

and you will then have

$$\$10.10 + \$0.101 = 1.01 \times \$10.10 = 1.01 \times (1.01 \times \$10) = \$10.201$$

in your account. Although the bank won't pay you the 0.1c, they leave it in for future calculations.

*Can you see a pattern?* At the start of each month, the new amount in your account will be the amount you had last month times 1.01. *Now can you answer the questions above? Write down a formula using exponential notation for the amount in your account after n months.*

## 3.4 Solutions

### Piles of Paper

1 sheet of paper = 0.1 mm thick.

After 1 cut, 2 ($2^1$) sheets of paper = 0.2 mm thick.
After 2 cuts, 4 ($2^2$) sheets of paper = 0.4 mm thick.
After 3 cuts, 8 ($2^3$) sheets of paper = 0.8 mm thick.
⋮
After 42 cuts, 4, 398, 046, 511, 104 ($2^{42}$) sheets of paper $\approx$ 439, 804, 651, 110 mm thick, about 439,805 kilometres — a little more than the distance from the Earth to the Moon.

### Shoeing a Horse

A horse needs 4 new horseshoes with 5 nails in each.

| Nail | Cost of Nail |
|---|---|
| 1st | 1c |
| 2nd | 2c |
| 3rd | 4c |
| 4th | 8c |
| 5th | 16c |
| 6th | 32c |
| 7th | 64c |
| 8th | $1.28 |
| 9th | $2.56 |
| 10th | $5.12 |
| ⋮ | ⋮ |
| 19th | $2,621.44 |
| 20th | $5,242.88 |
| Total | $10,485.75 |

If the man pays $100 per nail, it will cost him $2,000 to shoe the horse.

If he pays by the nail as in the table above (the total cost is the sum of all the costs in the second column), it will cost him $10,485.75, more than 5 times the flat rate. The choice is now obvious, and demonstrates clearly how rapidly exponential functions can increase.

**Interest Rates**

| **Deposit** | **Annual Interest Rate** | **Time Period** | **Value** |
|---|---|---|---|
| \$10.00 | 12% compounded monthly | after 1 year | \$11.27 (= \$10.00 $\times 1.01^{12}$) |
| | | after 5 years | \$18.17 (= \$10.00 $\times 1.01^{60}$) |
| | | after 10 years | \$33.00 (= \$10.00 $\times 1.01^{120}$) |

A formula for compound interest is therefore

$$D \times \left(1 + \frac{i}{100}\right)^n,$$

where $D$ is the initial deposit, $i\%$ the interest rate per month and $n$ the number of months the money has been left in the bank.

# 4 Population-Modelling Programs for Graphics Calculators

All programs (except POPMATX3/4) are available for the TI-83 family, the Casio CFX-9850 family and the Sharp EL-9650 family.

Download from *www.unsw.adfa.edu.au/pems/news/high_school/hsc_activities.html*

## AUSPOP — models the Australian population 1906 – 1996
## WORLDPOP — models the world population 1940 – 2000

Fits exponential, logistic and Australian Bureau of Statistics (ABS) or US Bureau of Statistics (USBS) curves to the population data and allows the models to be used for extrapolation into the future.

**Use:** Run the program. The screen tells you which keys can be used at any given time. After the program has finished, you can TRACE the graphs on the screen or manually replot the data and any of the models (Y1: exponential; Y2: logistic; Y3: ABS/USBS) by turning on/off the appropriate plot/functions in [Y=] (*TI*) / [Y=] or [STATPLOT] (*Sharp*) / [MENU] [5] or [MENU] [2] [F1] [F4] (*Casio*). You can also change the WINDOW.

Press [STAT] Edit (*TI, Sharp*) or [MENU] [2] (*Casio*) to see the data and model values.

## LOGISTIC — populations of bacteria or kangaroos

Sets up the graphics for a population of bacteria obeying the discrete logistic equation (Section 2.6) or for a population of kangaroos obeying the discrete logistic equation with culling (Section 2.7).

**Use:** Run the program. Select BACTERIA or KANGAROOS. Input the appropriate parameters at the prompts ($0 < u(0) < 1$ for the bacteria). The program plots population versus cycle number (time). Use the arrow keys to trace the graph or press [ENTER] / [EXE] to return to the main menu.

## MALTHUS/1/2 — models the US population 1790 – 1990

MALTHUS1 fits an exponential curve to the population for 1790 – 1830, the years for which Malthus had data. MALTHUS2 fits exponential and logistic curves to the population for 1790 – 1990, and allows the models to be used for extrapolation into the future. The TI-83 program MALTHUS contains both MALTHUS1 and MALTHUS2.

**Use:** As for AUSPOP/WORLDPOP.

## POP — population-matrix multiplication

Multiplies a column vector $\boldsymbol{v}$ (populations in different classes) by a matrix $\boldsymbol{T}$ (transition or population-projection matrix), displays the new $\boldsymbol{v}$ and the sum of the components of $\boldsymbol{v}$ (total population).

**Use:** Store the $3 \times 3$ matrix $\boldsymbol{T}$ in matrix A and the $3 \times 1$ column matrix $\boldsymbol{v}$ in matrix B. Run the program for the first step. Press ENTER / EXE repeatedly for subsequent steps. This simple program is easily modified for larger matrices.

## POPMATX3/4 — population-matrix modelling

**TI-83 family only.**

POPMATX3 calculates and plots as a function of cycle number (time) the numbers in three age classes of a population — called here juveniles, adults and seniors — which is modelled using a Leslie matrix or other population-projection matrix.

POPMATX4 does the same thing, but with one additional age class, yearlings, which comes before juveniles. See Brault and Caswell, *Pod-specific demography of killer whales*, Ecology 74, 1444–1454 (1993).

If $\boldsymbol{v_0}$ is the $3 \times 1$ or $4 \times 1$ matrix containing the initial population values and $\boldsymbol{P}$ is the $3 \times 3$ or $4 \times 4$ population-projection matrix, the population after 1 cycle is given by $\boldsymbol{v_1} = \boldsymbol{P}\boldsymbol{v_0}$, after 2 cycles by $\boldsymbol{v_2} = \boldsymbol{P}\boldsymbol{v_1} = \boldsymbol{P^2}\boldsymbol{v_0}$, and so on. The population after $n$ cycles is then $\boldsymbol{v_n} = \boldsymbol{P}\boldsymbol{v_{n-1}} = \boldsymbol{P^n}\boldsymbol{v_0}$.

**Use:** Put your $3\times3$ or $4\times4$ population-projection matrix $\boldsymbol{P}$ (Leslie matrix or equivalent) in matrix $A$, the initial populations in the three or four ages classes in the $3 \times 1$ or $4 \times 1$ matrix $B$.

Run the appropriate program and input the number of cycles for which you wish to calculate and plot the population. The program will do the calculations and plot the graphs of three populations — juveniles, adults and seniors — versus cycle number (time). The program pauses in TRACE mode so that you can analyse the graphs. The arrow keys allow you to move along a graph or from graph to graph. Which graph the cursor is on is indicated at the top left of the screen.

Pressing ENTER then gives a plot of the total population versus cycle number. Again you can TRACE this graph. Displayed on this graph is the largest eigenvalue $\lambda_m$ of the population-projection matrix. The long-term total population varies as $\lambda_m^t$, where $t$ is time. Therefore, if $\lambda_m > 1$, the population increases; if $\lambda_m = 1$, the population is steady; if $\lambda_m < 1$, the population decreases.

Pressing ENTER again ends the program.

Once a program has finished, you can replot and TRACE any or all of the graphs of juvenile population, adult population or total population versus cycle by selecting the appropriate plot in the Y= menu. You may need to change the WINDOW either manually in the WINDOW menu or by using ZOOM 9 (*ZoomStat*). To plot the graphs of the yearling or senior population, you will have to change the Ylist in one of the plots using STATPLOT (2nd Y=).

The data generated are stored in lists which can be accessed by pressing STAT Edit. Scrolling across will show the lists not initially on the screen.

Run the POPMCLR program to finish up. It deletes the matrices and lists used in/generated by the POPMATX programs, turns off the plots and resets the lists you will see when you press STAT Edit.

POPMATX3, POPMATX4 and POPMCLR are contained in the group file POPMATX.83g.